CSE 16 8-13-24

SETS: Due tonight 11:59 PM

Final: Thursday 11:15-1:00 PM

Today: finish 7.1

skip 7.2 (take CSE 107)

start 8.1

Theorem

Let $E \subseteq S$. Then $\overline{E} = S - E$ has probability

$$P(\overline{E}) = 1 - P(E)$$

Proof

$$1 = P(S) = P(E \cup \bar{E}) = P(E) + P(\bar{E})$$

since $E \cup \bar{E} = S$ since $E \cap \bar{E} = \phi$.

$$\therefore \quad P(\bar{E}) = 1 - P(E). \quad \blacksquare$$

Also: $P(E) = 1 - P(\bar{E})$

Ex. What is the Prob. that a 5-card Poker hand contains at least one ace?

$S = \{\text{all 5-card Poker hands}\}$

$$|S| = \binom{52}{5} = 2{,}598{,}960.$$

$E = \{$hands containing 1 or 2 or 3 or 4 or 5 aces$\}$

$\overline{E} = \{$hands cont. 0 aces$\}$

$$|\overline{E}| = \binom{48}{5} = 1{,}712{,}304$$

$$\therefore P(E) = 1 - P(\overline{E})$$

$$= 1 - \frac{1712304}{2598960} = .\underbrace{3411}5\ldots$$

Theorem

Given $E, F \subseteq S$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Proof

divide both sides of

$$|E \cup F| = |E| + |F| - |E \cap F|$$

by $|S|$. $\blacksquare$

Ex.

what is the prob. that a 5-card Poker hand contains a red ace?

$$E = \{\text{contains } AH\}$$

$$F = \{\text{contains } AD\}$$

$$E \cap F = \{\text{hand that contain both } AH, AD\}$$

$$|E| = |F| = \binom{51}{4}$$

$$|E \cap F| = \binom{50}{3}$$

$$\therefore P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= \frac{2\binom{51}{4} - \binom{50}{3}}{\binom{52}{5}} = .\underbrace{18476}\ldots$$

Another way:

note $\overline{E \cup F} = \{\text{not AH and not AD}\}$

$$|\overline{E \cup F}| = \binom{50}{5}$$

so

$$P(E \cup F) = 1 - P(\overline{E \cup F}) = 1 - \frac{\binom{50}{5}}{\binom{52}{5}}$$

check !!

Ex.

To win a lottery you must pick 7 correct #s out of 40.

$$|S| = \binom{40}{7}$$

$$|E| = 1$$

so $P(E) = \dfrac{1}{\binom{40}{7}} = (5.36)\cdot 10^{-8}$

In general:

$$P(\text{win}) = \frac{\#\text{ ways of winning}}{\#\text{ ways of playing}}$$

Ex.

In another state, lottery commision picks 12 #s out of 60. To win you must pick 6 out of these 12 #s.

$$P(\text{win}) = \frac{\binom{12}{6}}{\binom{60}{6}}$$

$$= \frac{924}{50{,}063{,}860} = (1.85)\cdot 10^{-5}$$

Ex.

what is the Prob. that a random string of length 10 from $\{a, \dots, z\}$ contains no repeated letters?

$S = \{\text{strings of len. 10 from } \{a, \dots, z\}\}$

$|S| = 26^{10}$

$E = \{\text{strs. of len. 10 with } \underline{\text{no}} \text{ repeated letters}\}$

$|E| = 26 \cdot 25 \cdot 24 \cdots 17 = \frac{26!}{(26-10)!} = \frac{26!}{16!}$

$\therefore P(E) = \frac{26!}{16! \cdot 26^{10}} = .1365\ldots$

Exercise

find a formula for the Prob. that a random string of len. k from an n-element alphabet contains no repeated letters.

note: if $k > n$, then Prob. is 0 by Pigeonhole Principle.

Exercise

Let $|A| = 10$, $|B| = 26$. Let $f: A \to B$ be chosen at random. What is the Prob. that f is injective?

8.1 Applications of recurrence relations

Ex.

Let B_n denote the # of bit strings of len. n that contain 2 consecutive zeros, i.e. that contain the substring '00'.

Find a recurrence for

$$(B_n)_{n=0}^{\infty}$$

Observe

$B_0 = 0 \quad \emptyset$

$B_1 = 0 \quad \emptyset$

$B_2 = 1 \quad \{00\}$

$B_3 = 3 \quad \{000, 001, 100\}$

We construct a bit str. of len. n containing '00' by performing exactly one of the following subtasks:

(1) form a bit str. of len. $n-1$ containing '00', append 1.

$$\underbrace{xx\cdots x}_{n-1}\,1 \quad : \; \#\text{ways} = B_{n-1}$$

(2) form a bit str. of len. $n-2$ containing '00, append 10.

$$\underbrace{xx\cdots x}_{n-2}\,10 \quad : \; \#\text{ways} = B_{n-2}$$

(3) form an arbitrary bit str. of len $n-2$, append 00

$$\underbrace{\overset{\text{any}}{x\,x\cdots x}}_{n-2}\,00 : \#\text{ways} = 2^{n-2}$$

By the sum Rule

$$\begin{cases} B_n = B_{n-1} + B_{n-2} + 2^{n-2} \\ B_0 = B_1 = 0 \end{cases}$$

$B_2 = 0 + 0 + 2^0 = 1$ ✓

$B_3 = 1 + 0 + 2^1 = 3$ ✓

$B_4 = 3 + 1 + 2^2 = 8$

$B_5 = 8 + 3 + 2^3 = 19$

$B_6 = 19 + 8 + 2^4 = 43$

Find the Prob. that a random bit string of len. 6 contains 2 consec. zeros: '00'.

$$|S| = 2^6 = 64$$

$$|E| = 43$$

$$\therefore P(E) = \frac{43}{64} = .6719\ldots$$

Exercise

check solution!

$$B_n = \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Hint: recall $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ are roots $x^2 - x - 1 = 0$, i.e. $x^2 = x + 1$.

Ex.

Find a recurrence for the # of bit strings of len. n containing '000'.

let C_n = # such bit strings.

(1) $\underbrace{\overset{000}{x \cdots\cdots x}}_{n-1}\,1$: # ways $= C_{n-1}$

(2) $\underbrace{\overset{000}{x \cdots\cdots x}}_{n-2}\,1\,0$: # ways $= C_{n-2}$

(3) $\underbrace{\overset{000}{x \cdots x}}_{n-3}\,1\,0\,0$: # ways $= C_{n-3}$

(4) $\underbrace{\overset{\text{any}}{x \cdots x}}_{n-3}\,0\,0\,0$: # ways $= 2^{n-3}$

Thus

$$\begin{cases} C_n = C_{n-1} + C_{n-2} + C_{n-3} + 2^{n-3} \\ C_0 = C_1 = C_2 = 0 \end{cases}$$

$$C_3 = 0 + 0 + 0 + 2^0 = 1$$

$$C_4 = 1 + 0 + 0 + 2^1 = 3$$

$$C_5 = 3 + 1 + 0 + 2^2 = 8$$

$$C_6 = 8 + 3 + 1 + 2^3 = 20$$

$$\vdots$$

Exercise (8.1 #14)

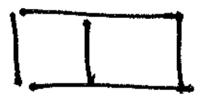
Find a recurrence for the # of ternary strings (alphabet = $\{0,1,2\}$) of length n containing '00'.

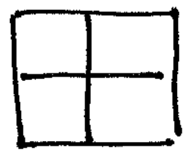
answer

$$\begin{cases} \overline{T}_0 = 0 \\ \overline{T}_1 = 0 \\ \overline{T}_n = 2\overline{T}_{n-1} + 2\overline{T}_{n-2} + 3^{n-2} \end{cases}$$

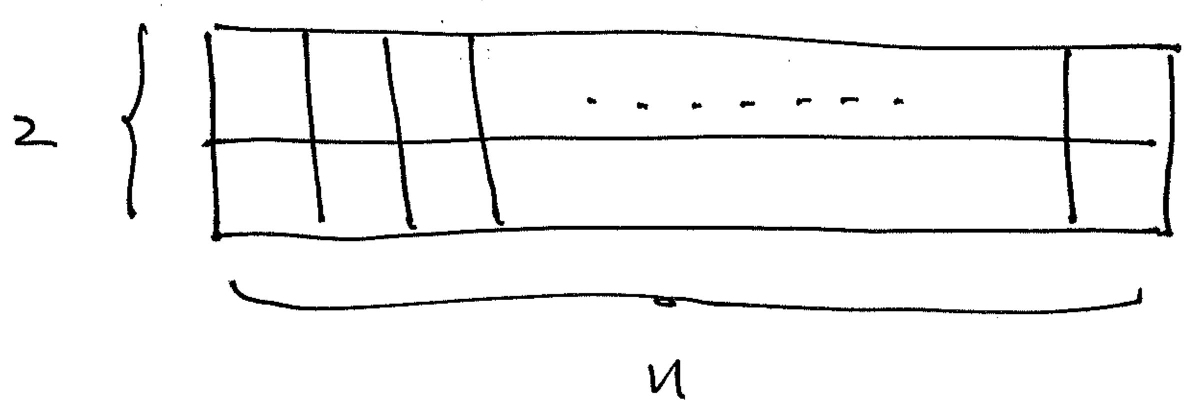
Ex.

We have 2 types of tiles:

1×2 

2×2 

Goal tile a $2 \times n$ walkway:



Find a recurrence for the # of such tilings.