CSE 16 8-1-24

- Quiz 4 Today: 12:35 – 1:00

6.3 Permutations & Combinations

Defn

A Permutation of a (finite) set is an ordered arrangement of its elements.

Ex. $S = \{1, 2, 3\}$

Permutations of S:

123, 132, 213, 231, 312, 321

Defn

A <u>k-permutation</u> of S is an ordered arrangement of k elements of S $(0 \le k \le |S|)$

Ex. $S = \{1, 2, 3\}$, $k = 0, 1, 2, 3$

	<u>k-Permutations</u>	<u>$P(3,k)$</u>
$k=0$:	ϕ (empty)	1
$k=1$:	1, 2, 3	3
$k=2$:	12, 21, 13, 31, 23, 32	6
$k=3$:	123, 132, 213, 231, 312, 321	6

Notation:

$P(n,k)$ = # of k-permutations of an n-set

Theorem: for $0 \le k \le n$

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Rmk

- follows from product rule.
- note if $k=n$, then a k-permutation is a permutation

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

<u>Defn</u>

A k-combination of S is an unordered collection of k of its elements, i.e. a k-element subset of S. $(0 \le k \le |S|)$.

<u>Ex</u> $S = \{1, 2, 3\}$ $k = 0, 1, 2, 3$

	k-combinations of S	$C(3,k)$
$k=0$:	ϕ	1
$k=1$:	$\{1\}, \{2\}, \{3\}$	3
$k=2$:	$\{1,2\}, \{1,3\}, \{2,3\}$	3
$k=3$:	$\{1,2,3\}$	1

Notation

- $C(n,k)$ = # of k-combinations of an n-set.
- other notation: $\binom{n}{k}$ read this 'n choose k'.
- also called: 'Binomial Coefficients'
- more notations: C^n_k, ${}^nC_k$, ${}_kC^n$

Theorem

for $0 \le k \le n$

$$C(n,k) = \frac{n!}{k!\,(n-k)!}$$

Proof

Let S be a set with $|S| = n$.

Consider the task of constructing a k-permutation of S.

We decompose this task into 2 subtasks as follows.

- choose a k-subset of S.
  $\# \text{ways} = C(n, k)$.
- choose a permutation of that k-set. $\# \text{ways} = k!$

By the product rule:

$$P(n, k) = C(n, k) \cdot k!$$

Thus

$$\frac{n!}{(n-k)!} = C(n,k)\cdot k!$$

$$\therefore \quad C(n,k) = \frac{n!}{k!\,(n-k)!} \quad \blacksquare$$

Ex How many bit strings of length 10 contain exactly 6 1's ?

1	2	3	4	5	6	7	8	9	10
	1		1	1		1	1		1

Positions occupied by 1's : $\{2,4,5,7,8,10\}$

a 6-subset of $\{1,2,3,4,5,6,7,8,9,10\}$

We must choose which 6 of the positions $\{1, 2, \ldots, 10\}$ are to be occupied by 1's.

$$\#\text{ways} = C(10,6) = \binom{10}{6}$$

$$\#\text{strings} = \binom{10}{6} = \frac{10!}{6!\,4!}$$

$$= \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = \boxed{210}$$

<u>note</u>: we could have chosen which 4 of 10 positions have 0's.

$$\#\text{strings} = \binom{10}{4} = \frac{10!}{4!\,6!} = \boxed{210}$$

Theorem

For $0 \le k \le n$

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof (algebraic)

$$RHS = \binom{n}{n-k}$$

$$= \frac{n!}{(n-k)!\,(\cancel{n}-(\cancel{n}-k))!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k} = LHS.$$

∎

2 types of proof:

- <u>algebraic</u>
  manipulate known identities

- <u>combinatorial</u>
  argue that LHS & RHS count the same set.

<u>alternative proof (combinatorial)</u>

Let $|S| = n$. Define for $0 \le k \le n$

$$S^{(k)} = \{k\text{-subsets of } S\} \subseteq \mathbb{P}(S)$$

Now define $f : S^{(k)} \to S^{(n-k)}$ by

$$f(A) = \overline{A} = S - A$$

observe $f \circ f(A) = \overline{\overline{A}} = A$

$\therefore$ $f \circ f = id_{S^{(k)}}$. $\therefore$ f is invertible with inverse $f^{-1} = f$. Thus f is a bijection.

$$\therefore \quad |S^{(k)}| = |S^{(n-k)}|.$$

$$\therefore \quad \binom{n}{k} = \binom{n}{n-k}. \quad \blacksquare$$

6.4 Binomial Coefficients

stick to: $\binom{n}{k}$ notation

observe

n	
0	$\binom{0}{0}=1$
1	$\binom{1}{0}=1,\ \binom{1}{1}=1$
2	$\binom{2}{0}=1,\ \binom{2}{1}=2,\ \binom{2}{2}=1$
3	$\binom{3}{0}=1,\ \binom{3}{1}=3,\ \binom{3}{2}=3,\ \binom{3}{3}=1$

$\vdots$

continuing

n													
0							1						
1						1		1					
2					1		2		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		15		6		1
⋮							⋮						⋱

Pascal's Triangle

Theorem (Pascal's Identity)

(1) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad (1 \leq k \leq n)$

alternate form

(2) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (1 \leq k < n)$

Proof of (1) (algebraic)

$$RHS = \binom{n}{k} + \binom{n}{k-1}$$

$$= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!}$$

$$= \frac{n!\,(n-k+1)}{k!\,(n-k+1)!} + \frac{n!\,k}{k!\,(n-k+1)!}$$

$$= \frac{n!\,\left((n-\cancel{k}+1)+\cancel{k}\right)}{k!\,\left((n+1)-k\right)!}$$

$$= \frac{(n+1)!}{k!\,\left((n+1)-k\right)!}$$

$$= \binom{n+1}{k} = \text{LHS}. \qquad \blacksquare$$