cse 16 7-9-24

- can an infinite set be a <u>subset</u> of a finite set?

  <u>no</u>

- can an infinite set be a <u>member</u> of a finite set?

  <u>yes</u> : $\{\mathbb{N}\}$

Theorem

If S is a finite set, then so is $\mathbb{P}(S)$ and

$$|\mathbb{P}(S)| = 2^{|S|}$$

Ex $S = \{1, 2, 3\}$

$$|\mathbb{P}(S)| = |\{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}|$$

$$= 8 = 2^3 = 2^{|S|}$$

Defn

An ordered collection of n elements is called an ordered n-tuple.

notation: $(x_1, x_2, x_3, \dots, x_n)$

$n=2$: (x_1, x_2) ordered pair

$n=3$: (x_1, x_2, x_3) " triple

$n=4$: (x_1, x_2, x_3, x_4) " 4-tuple

Two ordered n-tuples are equal iff they have same elements in same positions: $(x_1, x_2, x_3) = (y_1, y_2, y_3)$ iff $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$.

<u>Defn</u>

The <u>Cartesian product</u> of sets A, B is the set of all ordered pairs (x, y) with $x \in A$, $y \in B$.

$$A \times B = \{(x,y) \mid x \in A \wedge y \in B\}$$

<u>EX</u> $A = \{1, 2, 3\}$, $B = \{3, 4\}$. Then

$$A \times B = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$$

$$B \times A = \{(3,1), (3,2), (3,3), (4,1), (4,2), (4,3)\}$$

In general $A \times B \neq B \times A$

In general the Cartesian Product of n sets $A_1, A_2, \ldots, A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i \text{ for } 1 \le i \le n\}.$$

If each A_i is finite, then

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$$

In Particular,

$$|A \times B| = |A| \cdot |B|$$

Russell's Paradox

Let $\mathcal{S} = \{\text{all sets}\}$. Observe $\mathcal{S} \in \mathcal{S}$, but $\phi \notin \phi$. Define

$$\mathcal{R} = \{A \in \mathcal{S} \mid A \notin A\}.$$

Either $\mathcal{R} \in \mathcal{R}$ or $\mathcal{R} \notin \mathcal{R}$. But

$$\mathcal{R} \in \mathcal{R} \rightarrow \mathcal{R} \notin \mathcal{R}. \quad \times$$

$$\mathcal{R} \notin \mathcal{R} \rightarrow \mathcal{R} \in \mathcal{R}. \quad \times$$

<u>EX</u> another problem set:

$$A = \{B\}, \quad B = \{A\}$$

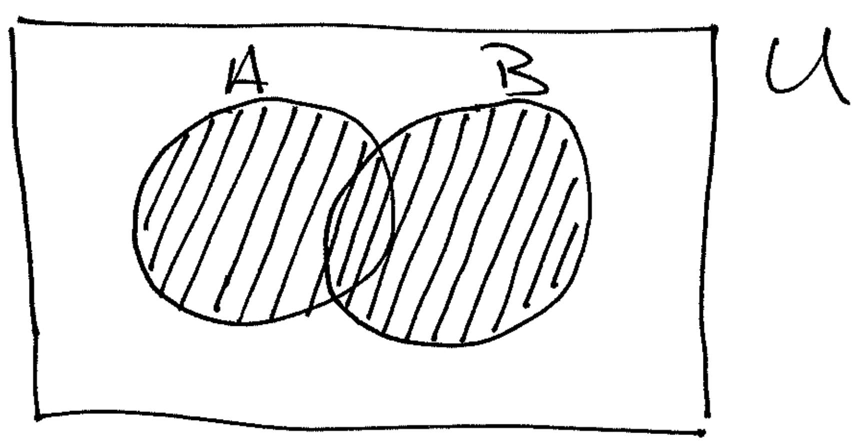
2.2 Set operations

Defn

The union of two sets A, B (where $A \subseteq U, B \subseteq U$) is

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$

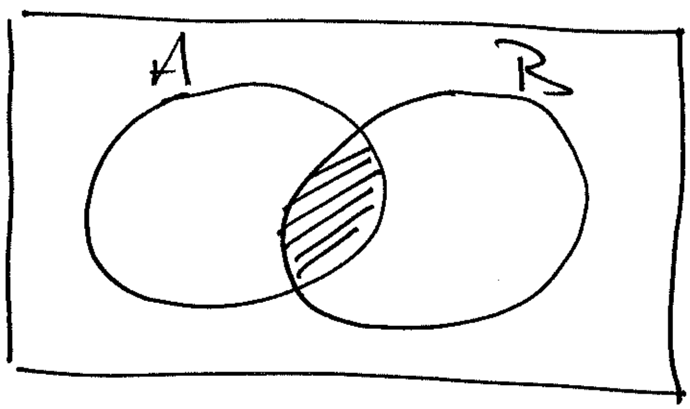
Venn diagram:



Defn

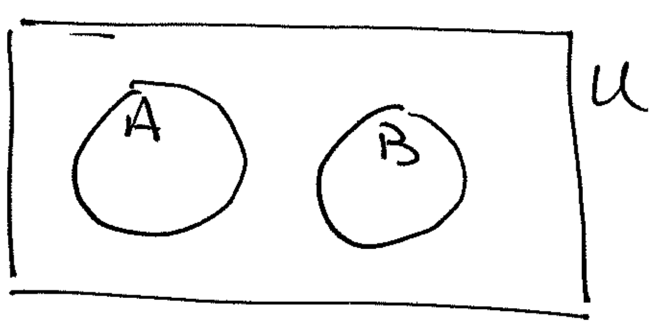
The <u>intersection</u> of A, B is

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$



Defn

We say A and B are disjoint iff $A \cap B = \emptyset$.

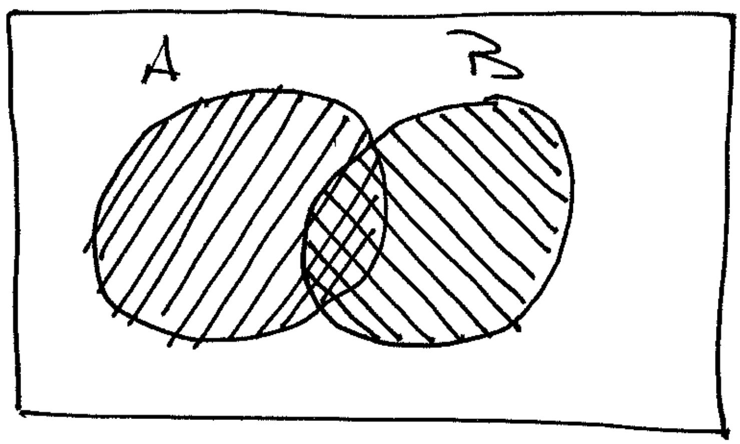


Theorem

If A, B are finite, then so are $A \cup B$ and $A \cap B$, and

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Picture!

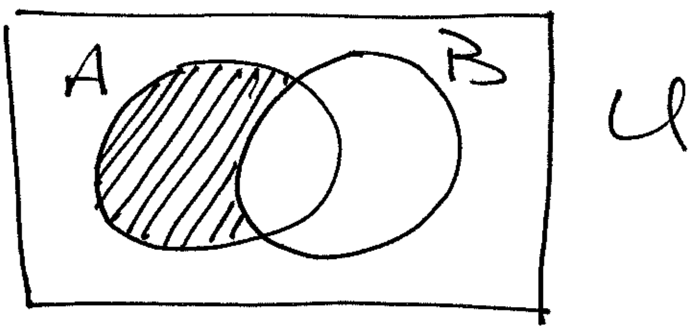


This is a special case of the Inclusion-Exclusion Principle (PIE)

<u>Defn</u>

The <u>set difference</u> $A-B$ is

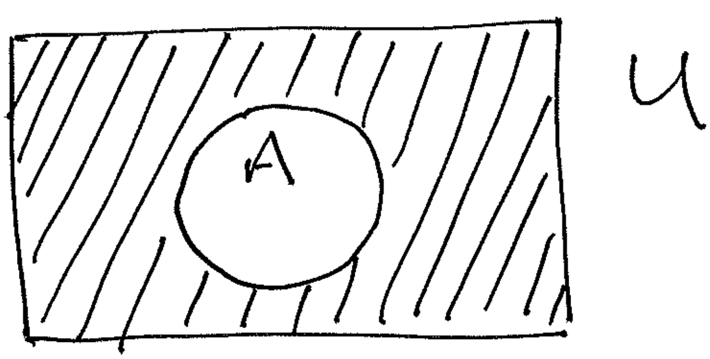
$$A-B=\{x\in U \mid x\in A \wedge x\notin B\}$$



<u>Defn</u>

The <u>complement</u> of A is the set

$$\overline{A}=U-A=\{x\in U \mid x\notin A\}$$



notice: $A - B = A \cap \overline{B}$

$$B - A = B \cap \overline{A}$$

$$\therefore \quad A - B \neq B - A$$

Read set Identities (P.130 table 1)

Ex,

Prove 1st DeMorgan

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- logical identities
- membership tables

Proof (using logical DeMorgan)

$$\overline{A \cap B} = \{x \in U \mid x \notin A \cap B\}$$

$$= \{x \in U \mid \neg x \in (A \cap B)\}$$

$$= \{x \in U \mid \neg (x \in A \wedge x \in B)\}$$

$$= \{x \in U \mid \neg x \in A \vee \neg x \in B\}$$

$$= \{x \in U \mid x \notin A \vee x \notin B\}$$

$$= \{x \in U \mid x \in \overline{A} \vee x \in \overline{B}\}$$

$$= \overline{A} \cup \overline{B}$$

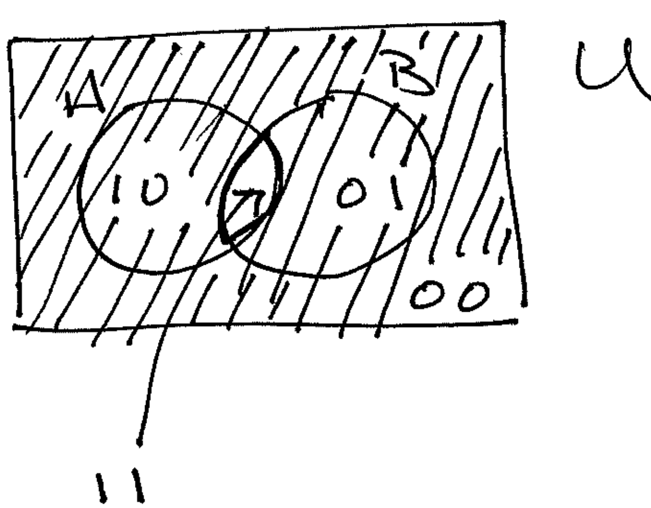
$\blacksquare$

Proof (membership table)

1 = membership

0 = non-membership

A	B	$A \cap B$	$\overline{A \cap B}$	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0



$\therefore$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Defn

The <u>symmetric Difference</u> of

A, B is

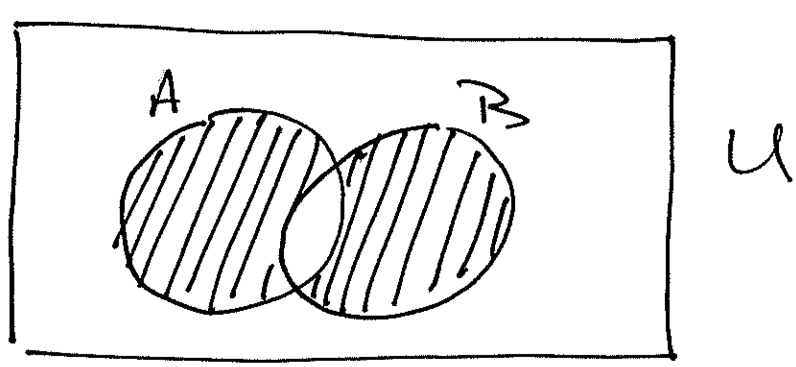
exor

$$A \oplus B = \{x \in U \mid x \in A \oplus x \in B\}$$

symm. diff.

Exercise

- $A \oplus B = (A - B) \cup (B - A)$
- $A \oplus B = (A \cup B) - (A \cap B)$



Representation of sets

Let $U = \{x_1, x_2, \dots, x_n\}$. Represent a subset $S \subseteq U$ by a bit-string where 1 = membership, 0 = non-membership.

Ex. $U = \{1, 2, 3, 4\}$

$$\{1,3,4\} \longleftrightarrow 1011$$

$$\{2,3\} \longleftrightarrow 0110$$

$$\{4\} \longleftrightarrow 0001$$

$$\phi \longleftrightarrow 0000$$

$$U \longleftrightarrow 1111$$

In general $S \leftrightarrow b$ where

$$\begin{cases} x_i \in S & \text{iff} \quad b_i = 1 \\ x_i \notin S & \text{iff} \quad b_i = 0 \end{cases}$$

we have operations

union $\longleftrightarrow$ bitwise or $\vee$

intersection $\longleftrightarrow$ bitwise and $\wedge$

symm. diff. $\longleftrightarrow$ bitwise exor $\oplus$

Ex. $\{1,3,4\} \oplus \{2,3\} = \{1,2,4\}$

$$(1011) \oplus (0110) = (\underset{\underset{1}{\uparrow}}{1}\ \underset{\underset{2}{\uparrow}}{1}\ \underset{\underset{3}{\uparrow}}{0}\ \underset{\underset{4}{\uparrow}}{1})$$

Notation

Given sets $A_1, A_2, A_3, \ldots\ldots$

- $A_1 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$

- $A_1 \cup A_2 \cup \cdots\cdots = \bigcup_{i=1}^{\infty} A_i$

- $A_1 \cap A_2 \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

- $A_1 \cap A_2 \cap \cdots\cdots = \bigcap_{i=1}^{\infty} A_i$

2.3 Functions

Defn

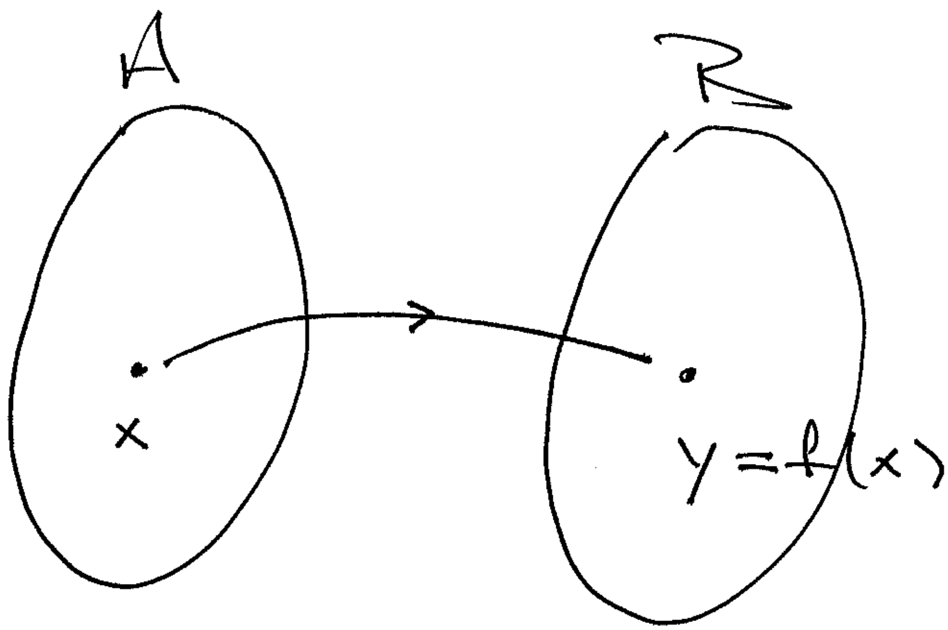
A function (map, mapping, transformation) consists of

(1) a set A called its Domain

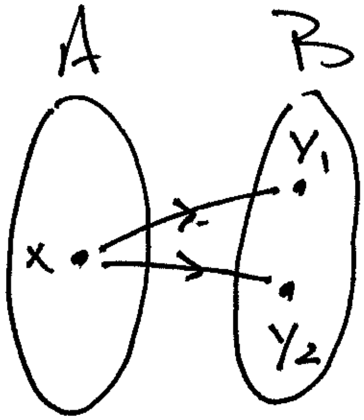
(2) a set B called its Codomain

(3) a rule f that assigns to each $x \in A$ a unique $y \in B$. Write $y = f(x)$, and call y the image of x under f, and x a pre-image of y under f.

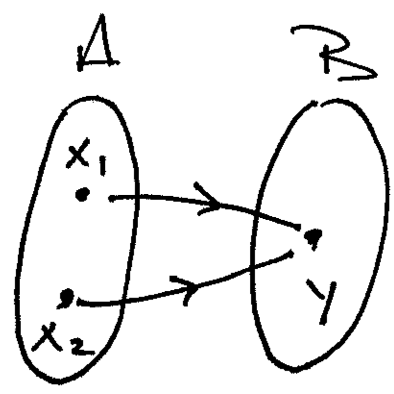
notation: $f : A \to B$



Uniqueness



not a function



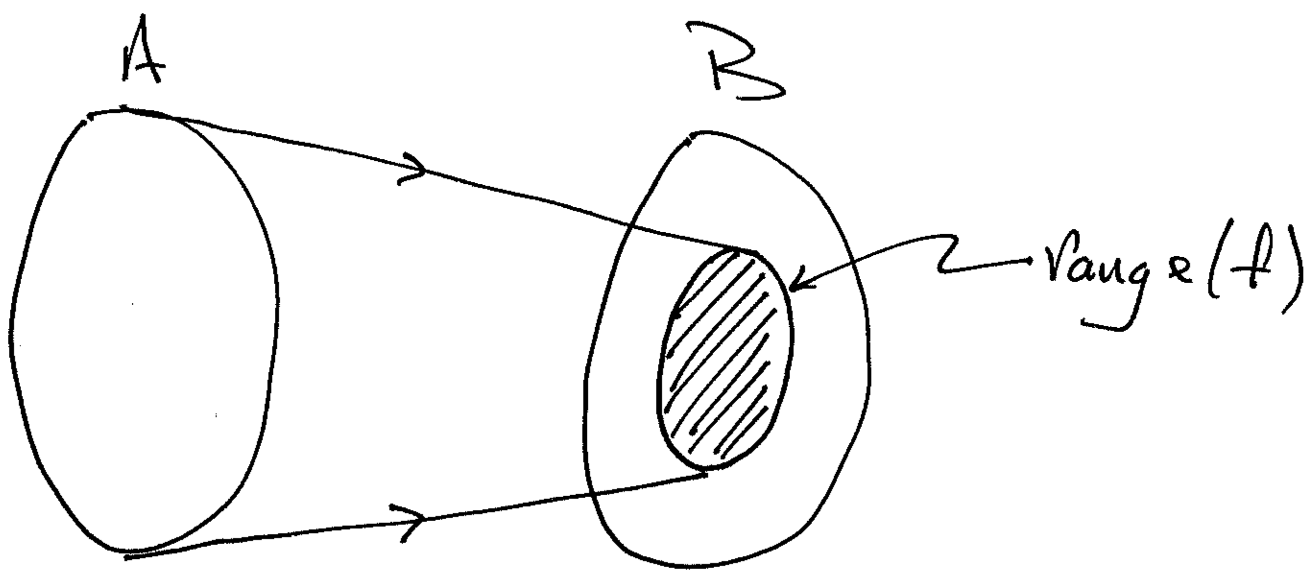
OK, this is a function.

Defn

The <u>range</u> of $f : A \to B$ is the set

$$\text{range}(f) = \{ y \in B \mid \exists x \in A : y = f(x) \}$$

$$= \{ f(x) \mid x \in A \}$$



Ex. $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

$$\text{range}(f) = \{0, 1, 4, 9, 16, \ldots\}$$

Ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$$\text{range}(f) = [0, \infty)$$

$$= \{y \in \mathbb{R} \mid y \geq 0\}$$