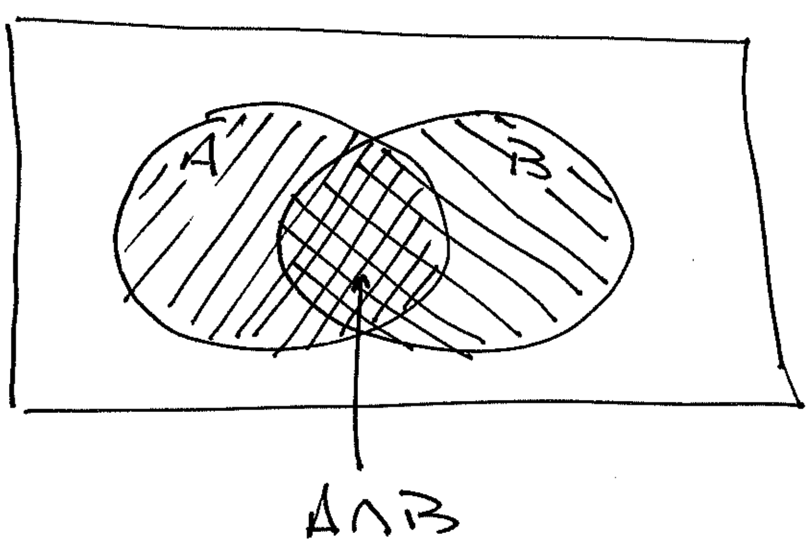
CSE 16 7-31-24

- Quiz 4: Tomorrow 8/1

  Cover: 5.1, 5.2, 5.3, 6.1

PIE (Principle of Inclusion Exclusion)

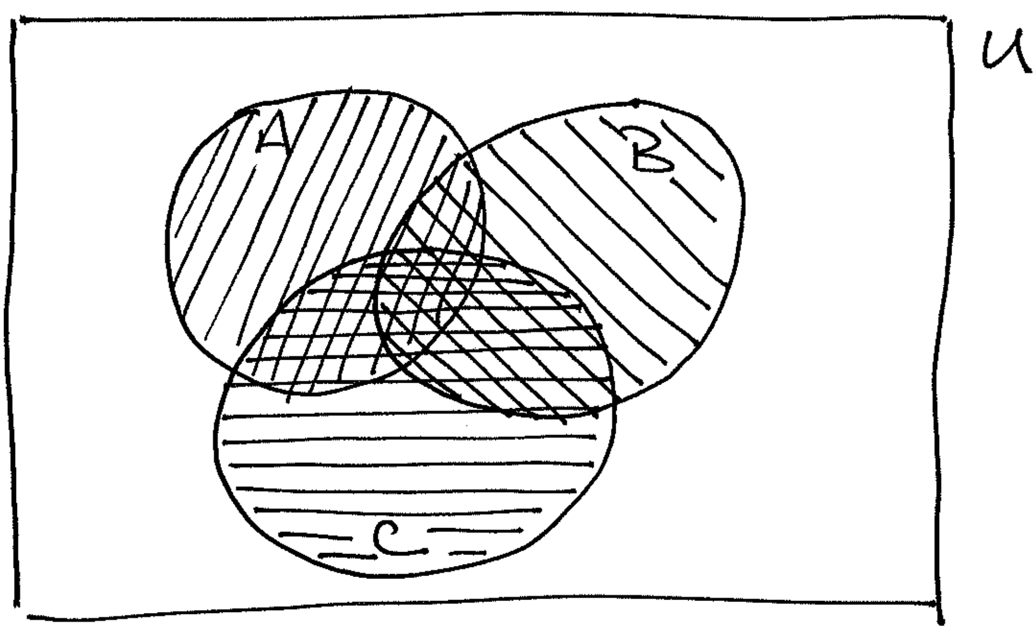
- $|A \cup B| = |A| + |B| - |A \cap B|$



- $|A \cup B \cup C| = \left(|A| + |B| + |C|\right)$

$$- \left(|A \cap B| + |A \cap C| + |B \cap C|\right)$$

$$+ |A \cap B \cap C|$$



- $|A \cup B \cup C \cup D|$

$$= \left(|A| + |B| + |C| + |D|\right)$$

$$- \left(|A\cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|\right)$$

$$+ \left(|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|\right)$$

$$- |A \cap B \cap C \cap D|$$

n-set General PIE

- $$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

$$- \sum_{i<j} |A_i \cap A_j|$$

$$+ \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$

$$\vdots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Defn

The <u>Euler Totient Function</u>

$\phi(n)$ is the number of numbers in $\{1, 2, \ldots, n\}$ that are relatively prime to n.

<u>Ex.</u> Find $\phi(1000)$

$\phi(1000) = 1000 - $ (# of # <u>not</u> rel. Prime to 1000)

A number is not relatively prime to $1000 = 2^3 \cdot 5^3$ iff it is divisible by 2 or by 5 (or possibly both)

(# of #s div. by 2) $= \frac{1000}{2} = 500$

(# of #s div. by 5) $= \frac{1000}{5} = 200$

(# of #s div by 10) $= \frac{1000}{10} = 100$

Therefore

$$\phi(1000) = 1000 - (500 + 200 - 100) = \boxed{400}$$

Exercise

- let $n = p^k q^m$ where p, q prime

show

$$\phi(p^k q^m) = p^k q^m - (p^{k-1} q^m + p^k q^{m-1} - p^{k-1} q^{m-1})$$

$\vdots$ (algebra)

$$= (p^k - p^{k-1})(q^m - q^{m-1})$$

$$= \phi(p^k) \cdot \phi(q^m)$$

- let $n = p^k \cdot q^m \cdot r^l$ where p, q, r prime.

find a formula for

$$\phi(n) = \phi(p^k q^m r^l)$$

6.2 Pigeonhole Principle (P.P.)

Theorem (P.P.)

If $K+1$ (or more) objects are placed in K boxes, then (at least) 1 box contains (at least) 2 objects.

Equivalently:
if $f: A \to B$ where A, B are finite and $|A| > |B|$, then f is not injective.

Ex.

A drawer contains 12 black and 12 brown socks. a man removes socks in the dark.

what is the smallest # of socks that most be removed to guarantee that he has 2 of same color.

objects ~ socks

boxes ~ colors

black brown

answer: $\boxed{3}$

floor & ceiling functions

- floor: $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$
- ceiling: $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$

are defined as

- $\lfloor x \rfloor$ = greatest int less than or equal to x
- $\lceil x \rceil$ = least int greater than or equal to x.

Equivalently:

- $n = \lfloor x \rfloor$ iff $n \leq x < n+1$
- $n = \lceil x \rceil$ iff $n-1 < x \leq n$

Equivalently: $\lfloor x \rfloor$ and $\lceil x \rceil$ are the unique integers satisfying

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

<u>Theorem</u> (Generalized P.P. or G.P.P.)

If n (or more) objects are placed in k boxes, then (at least) 1 box contains (at least) $\lceil \frac{n}{k} \rceil$ objects.

<u>note</u>: if $n = k+1$ we recover the P.P. since

$$\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{k+1}{k} \right\rceil = \left\lceil 1 + \frac{1}{k} \right\rceil = 2 .$$

Proof (contradiction)

Assume each box contains at most $\left(\lceil \frac{n}{k} \rceil - 1\right)$ objects. Then

$$n \le k \cdot \left(\lceil \tfrac{n}{k} \rceil - 1\right) < \cancel{k}\left(\left(\tfrac{n}{\cancel{k}} + \cancel{1}\right) - \cancel{1}\right) = n$$

$\therefore$ $n < n$ $\cdot\!\!\times\!\!\cdot$

This $\cdot\!\!\times\!\!\cdot$ shows our assumption was false, so some box contains at least $\lceil \frac{n}{k} \rceil$ objects. $\blacksquare$

Ex. Sock drawer problem.

what is the smallest # socks whose removal would guarantee 6 of same color.

<u>solution</u>

let answer be n.

objects ~ socks : n

boxes ~ colors : 2

we seek smallest n s.t.

$$\left\lceil \frac{n}{2} \right\rceil = 6$$

i.e. $$5 < \frac{n}{2} \le 6$$

i.e. $$10 < n \le 12$$

$\therefore$ $\boxed{n = 11}$.

Ex.

How many students must be enrolled at a university to guarantee that at least 100 were born in the same month.?

objects ~ students : n

boxes ~ months : 12

We seek the smallest n s.t.

$$\left\lceil \frac{n}{12} \right\rceil = 100$$

$$\underbrace{99 < \frac{n}{12}} \leq 100$$

$$\therefore \quad n = 12 \cdot 99 + 1 = \boxed{1189}$$

Ex.

There are 38 different periods in which to schedule classes at a University. There are 677 classes. How many rooms are necessary?

objects ~ classes : 677

boxes ~ periods : 38

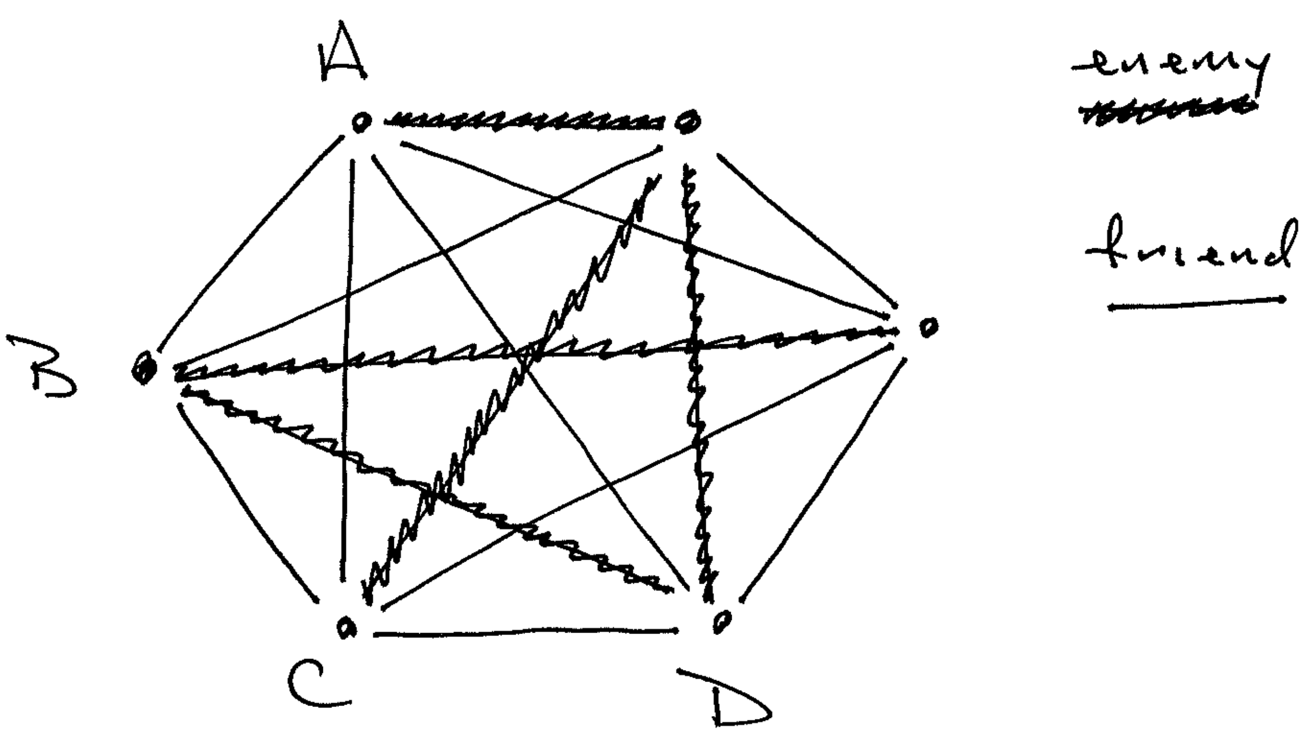
By the GPP at least 1 period has

$$\left\lceil \frac{677}{38} \right\rceil = \lceil 17.8 \rceil = 18$$

classes assigned to it, so at least $\boxed{18}$ rooms are needed.

Theorem

In any group of 6 People in which any pair are either friends or enemies, there must exist either 3 mutual friends or 3 mutual enemies.



called K_6

Proof

let A be a person in the group.

The remaining 5 people fall into 2 classes:

$$F = \{\text{friends of } A\}$$

$$E = \{\text{enemies of } A\}$$

By the GPP, at least $\lceil \frac{5}{2} \rceil = 3$ are in the same class.

Case 1: $|F| \geq 3$

say B, C, D are friends of A.

If any 2 of these are friends (of each other), say B and C, then A, B, C are <u>3 mutual friends</u>. otherwise B, C, D are <u>3 mutual enemies</u>.

<u>Case 2</u>: $|E| \geq 3$.

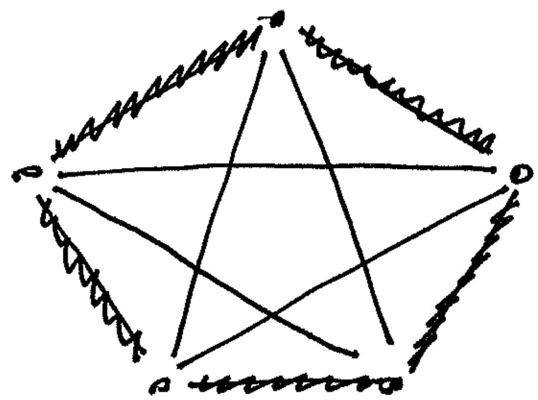
follows from case 1 by swapping 'friend' and 'enemy'.

$\blacksquare$

<u>Note</u>:

6 is the smallest group for which this conclusion is true.

Ex. 5 People



K_5

$R(3,3) = 6$

$R(m,\ell) = n$ iff n is

smallest # s.t. any 2-coloring of K_n contains either a monochrom. K_m or K_ℓ.