CSE 16 7-30-24

5.3 Recursion

Ex. Recall

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \quad (n \geq 2) \end{cases}$$

Thm

$$\forall n \geq 1 : \boxed{\sum_{k=1}^{n} F_{2k-1} = F_{2n}} \leftarrow P(n)$$

I. $P(1)$ says: $F_1 = F_2$, i.e. $1 = 1$ ✓.

II b. $\forall n > 1 : P(n-1) \Rightarrow P(n)$

Let $n > 1$, i.e. $n \geq 2$. Assume

$$\sum_{k=1}^{n-1} F_{2k-1} = F_{2n-2}$$

We must show

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n} .$$

So

$$\sum_{k=1}^{n} F_{2k-1} = \left(\sum_{k=1}^{n-1} F_{2k-1}\right) + F_{2n-1}$$

$$= F_{2n-2} + F_{2n-1} \quad \Big\} \text{ by the ind. hyp.}$$

$$= F_{2n} \quad \Big\} \text{ by the Fib. rec.}$$

Result follows by 1^{st} PMI. ∎

Ex $\forall n \geq 1 :$ $\boxed{\sum_{k=1}^{n} F_{k-1} F_k = \begin{cases} F_n^{\,2} & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases}}$

$P(n)$

Proof:

I. $P(1)$ says $F_0 \cdot F_1 = F_0 \cdot F_2$,

i.e. $0 = 0$, which is true ✓

IIa. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$. Assume

$$\sum_{k=1}^{n} F_{k-1} F_k = \begin{cases} F_n^{\,2} & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases}$$

we must show

$$\sum_{k=1}^{n+1} F_{k-1} F_k = \begin{cases} F_{n+1}^2 & (n \text{ odd}) \\ F_n F_{n+2} & (n \text{ even}) \end{cases}$$

so

$$\sum_{k=1}^{n+1} F_{k-1} F_k = \left(\sum_{k=1}^{n} F_{k-1} F_k \right) + F_n F_{n+1}$$

$$= \left\{ \begin{matrix} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{matrix} \right\} + F_n F_{n+1}$$

$$= \begin{cases} F_n^2 + F_n F_{n+1} & (n \text{ even}) \\ F_{n-1} F_{n+1} + F_n F_{n+1} & (n \text{ odd}) \end{cases}$$

$$= \begin{cases} F_n(F_n + F_{n+1}) & (n \text{ even}) \\ (F_{n-1} + F_n) F_{n+1} & (n \text{ odd}) \end{cases}$$

$$= \begin{cases} F_n \cdot F_{n+2} & (n \text{ even}) \\ F_{n+1} \cdot F_{n+1} & (n \text{ odd}) \end{cases}$$

$$= \begin{cases} F_{n+1}^2 & (n \text{ odd}) \\ F_n F_{n+2} & (n \text{ even}) \end{cases}$$

Result follows by 1st P.M.I. ∎

Exercise

$$\forall n \geq 1 : \quad GCD(F_n, F_{n+1}) = 1$$

6.1 Counting

Defn

A string is a finite sequence of symbols from some alphabet S, usually $|S| < \infty$

Ex

how many strings of length 2 are there from $S = \{a, b, c, \dots, z\}$

$$26 \left\{ \underbrace{\begin{array}{ccccc} aa & ab & ac & \dots & az \\ ba & bb & bc & \dots & bz \\ \vdots & \vdots & \vdots & & \vdots \\ za & zb & zc & \dots & zz \end{array}}_{26} \right.$$

$$\#\text{strings} = 26 \cdot 26 = 26^2 = \boxed{676}$$

<u>Ex</u> how many such strings of length 5?

$$\begin{array}{lccccc} & \underline{26} \cdot & \underline{26} \cdot & \underline{26} \cdot & \underline{26} \cdot & \underline{26} \\ \text{Position:} & 1 & 2 & 3 & 4 & 5 \end{array}$$

$$\#\text{strings} = 26^5 = \boxed{11,881,376}$$

<u>Ex</u> how many such strings have no repeated letters?

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 21 = \boxed{7,893,600}$$

Ex. How many bit strings of length K are there $(K \geq 0)$

alphabet $= \{0, 1\}$

$$\underbrace{2 \cdot 2 \cdot 2 \cdot \cdots 2}_{K \text{ bits}} = \boxed{2^K}$$

Ex. How many strings of len. K from an alphabet of size n are there

$$\begin{array}{lccccc} \# \text{choices} & \underline{n} \cdot & \underline{n} \cdot & \underline{n} \cdot & & \cdot \underline{n} \\ \text{Positions}: & 1 & 2 & 3 & \cdots & K \end{array}$$

$$\boxed{\# \text{strings} = n^K}$$

Ex. how many such strings have no repeated letters?

#choices: $\underline{n}\cdot\underline{(n-1)}\cdot\underline{(n-2)}\cdots\underline{(n-k+1)}$
Position: 1 2 3 ... k

$$\boxed{\#\text{strings} = n\cdot(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}}$$

note:

$$\#\text{strings w.o. repetion} = \begin{cases} \dfrac{n!}{(n-k)!} & 0 \le k \le n \\ 0 & k > n \end{cases}$$

note: $k=0$ gives 1 string {empty string}.

Product Rule

Suppose a task T is decomposed into k subtasks $T_1, T_2, \ldots, T_k$ in such a way that after

$$T_1, T_2, \ldots, T_{i-1}$$

have been performed, the T_i can be performed in n_i ways $(1 \le i \le k)$. Then T can be performed in

$$n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

ways.

Theorem

If A, B are finite sets, then so is $A \times B$, and

$$|A \times B| = |A| \cdot |B|$$

Proof

Task T is to form a pair $(x, y) \in A \times B$. Subtasks

	# ways
T_1 : choose $x \in A$:	$\|A\|$
T_2 : choose $y \in B$:	$\|B\|$

By the prod. rule T can be performed in $|A| \cdot |B|$ ways $\therefore |A \times B| = |A| \cdot |B|$. ∎

Theorem

If S is a finite set, then so is $\mathbb{P}(S)$ and

$$|\mathbb{P}(S)| = 2^{|S|}$$

Proof.

Task T is to construct a subset

$$A \subseteq S = \{x_1, x_2, \ldots, x_n\}$$

assuming $|S| = n$. Then T decomposes!

	<u>choose</u>	<u>$\exists$ ways</u>
T_1 :	$x_1 \in A$ or $x_1 \notin A$	2
T_2 :	$x_2 \in A$ or $x_2 \notin A$	2
$\vdots$		$\vdots$
T_n :	$x_n \in A$ or $x_n \notin A$	2

By the product rule of performing T is $2 \cdot 2 \cdots 2 = 2^n$, hence

$$|\mathbb{P}(S)| = 2^n = 2^{|S|}$$

$\blacksquare$

Defn

Let A, B be sets. The set of functions with domain A and codomain B, $f : A \to B$, is denoted

$$B^A = \{ f \mid f : A \to B \}$$

Theorem

If A, B are finite sets, then so is B^A, and

$$|B^A| = |B|^{|A|}$$

Proof

Task T is to construct a function $f: A \to B$. Suppose $|A| = k$ and $|B| = n$, and

$$A = \{x_1, x_2, \ldots, x_k\}$$

T decomposes into subtasks

	choose	# ways
T_1:	$f(x_1) \in B$	n
T_2:	$f(x_2) \in B$	n
$\vdots$		$\vdots$
T_k:	$f(x_k) \in B$	n

$$\therefore \; |B^A| = n \cdot n \cdots n = n^k = |B|^{|A|}$$

∎

Exercise

Show that the number of injective functions $f: A \to B$ where $|A| = k$ and $|B| = n$ is

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Sum Rule

Suppose a task T can be performed by exactly one of the subtasks

$$T_1, T_2, \ldots, T_k$$

where T_i can be performed in n_i ways $(1 \leq i \leq n)$

Then T can be performed in

$$n_1 + n_2 + \cdots + n_k$$

ways.

Theorem

Let $A_1, A_2, \ldots, A_k$ be finite sets that are pairwise disjoint, i.e. ($A_i \cap A_j = \phi$ for any $i \neq j$). Then

$$S = A_1 \cup A_2 \cup \cdots \cup A_k$$

is also finite, and

$$|S| = |A_1| + |A_2| + \cdots + |A_k|$$

Ex.

how many strings of length at most 5 (i.e. len $= 0, 1, 2, .., 5$) can be formed from an alphabet of size 12 ?

		# ways
T_5 :	choose a str. of len. 5 :	12^5
T_4 :	" " " " " 4 :	12^4
T_3 :	" - - " " 3 :	12^3
T_2 :	- - - - - 2 :	12^2
T_1 :	- - - - 1 :	12^1
T_0 :	- - - - 0 :	12^0

By the sum rule, there are

$$12^0 + 12^1 + 12^2 + \cdots + 12^5 = \frac{12^6 - 1}{12 - 1}$$

$$= \boxed{271{,}453}$$

strings of length at most 5.

Principle of Inclusion/Exclusion (PIE)

- If A, B are finite so are $A \cup B$ and $A \cap B$ and

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Special case: $A \cap B = \emptyset$

  sum rule

$$|A \cup B| = |A| + |B|$$

Ex. how many bit strings of length 7 either begin with '00' or end with '111'?

Solution

let $A = \{00xxxxx\}$ $|A| = 2^5$

$B = \{xxxx111\}$ $|B| = 2^4$

$A \cap B = \{00xx111\}$ $|A \cap B| = 2^2$

$\therefore |A \cup B| = 2^5 + 2^4 - 2^2 = \boxed{44}$

General PIE

- $|A_1 \cup A_2 \cup A_3|$

$$= \left(|A_1| + |A_2| + |A_3|\right)$$

$$- \left(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\right)$$

$$+ |A_1 \cap A_2 \cap A_3|$$

... more to come ...