CSE 16 7-25-24

- Quiz 3 Today 12:35 – 1:00.

- weak induction (1st PMI)

  I. $P(1)$

  IIa. $\forall n \geq 1 : P(n) \to P(n+1)$

  IIb. $\forall n > 1 : P(n-1) \to P(n)$

  Conclude: $\boxed{\forall n \geq 1 : P(n)}$

  Another variation:

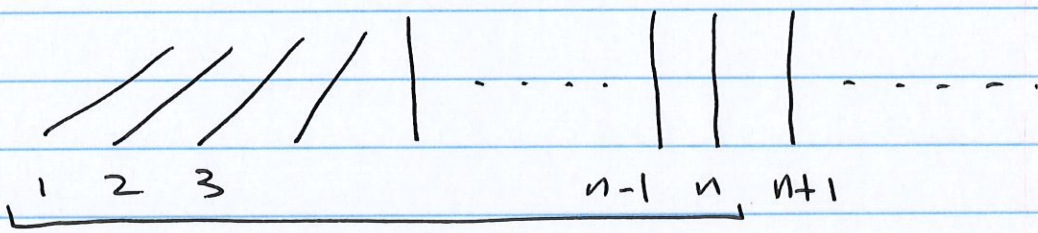
- Strong induction (2nd PMI).

  Also has 2 Parametrizations

I. $P(1)$

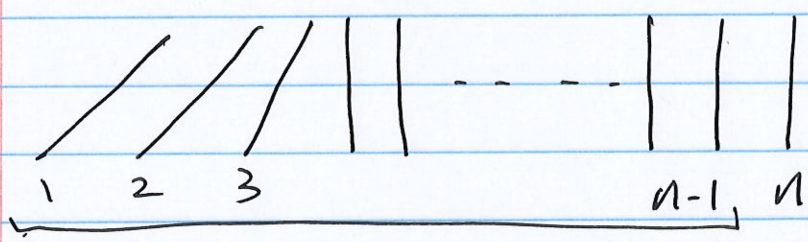
IIc. $\forall n \geq 1 : \underbrace{(P(1) \wedge \ldots \wedge P(n))}_{\text{induction hyp.}} \rightarrow P(n+1)$

Domino analogy:



IId. $\forall n > 1 : \underbrace{(P(1) \wedge \ldots \wedge P(n-1))}_{\text{ind. hyp.}} \rightarrow P(n)$

Domino analogy



We use form IId.

Theorem
Every positive integer $n \in \mathbb{Z}^+$ can be expressed as a product of (zero or more) primes.

Rmk
- This is half of FTA
- The other half is uniqueness see sec. 4.3 for that.

Proof
Let $P(n) =$ ' n is a product of primes'

I. $P(1)$ is true since 1 is the empty product.

IId. $\forall n > 1 : (P(1) \wedge \ldots \wedge P(n-1)) \to P(n)$

Let $n > 1$ be chosen arbitrarily.

Assume for any k in the range $1 \le k < n$, that k is a product of primes. } ind. hyp.

We must show that n is a product of primes. } ind conc.

Case 1: n is itself prime.
In this case n is a product of (just one) prime.

Case 2: n is composite.
In this case

$$n = a \cdot b$$

for some $1 < a < n$ and $1 < b < n$.
By the induction hypothesis, both a and b are products of primes, say

$$a = p_1 \cdot p_2 \cdots p_\ell$$

and

$$b = q_1 \cdot q_2 \cdots q_m$$

But then

$$n = a \cdot b = p_1 \cdots p_\ell \cdot q_1 \cdots q_m$$

is a product of primes.

Result follows for all $n \geq 1$ by 2^{nd} PMI.

∎

We use strong induction when the domino(s) that topple the n^{th} domino is anything but the immediately preceding domino.

Sometimes multiple base cases are necessary.

Ex. $\forall n \geq 2 : \boxed{\exists x \geq 0, \exists y \geq 0 : n = 3x + 2y}$

$P(n)$

Proof

I Two base cases

$P(2)$: $2 = 3 \cdot 0 + 2 \cdot 1$ ✓
$P(3)$: $3 = 3 \cdot 1 + 2 \cdot 0$ ✓

II Id. $\forall n > 3 : (P(2) \wedge \ldots \wedge P(n-1)) \Rightarrow P(n)$

highest base case   lowest base case

Let $n > 3$, i.e. $n \geq 4$ be arbitrary.

ind. hyp. $\left\{ \begin{array}{l} \text{Assume for all } k \text{ in range } 2 \leq k < n \\ \text{that} \\ \quad \exists x' \geq 0, \exists y' \geq 0 : k = 3x' + 2y' \end{array} \right.$

ind. conc. { we must show that

$$\exists x \geq 0, \exists y \geq 0 : n = 3x + 2y$$

Since $n \geq 4$ we have

$$2 \leq n-2 < n$$

By the induction hypothesis, there exist $x' \geq 0$ and $y' \geq 0$ such that

$$n-2 = 3x' + 2y'$$

Let $x = x'$ and $y = y'+1$. Then $x \geq 0$, and $y \geq 0$, and

$$\begin{aligned} n &= (n-2) + 2 \\ &= (3x' + 2y') + 2 \\ &= 3x' + 2(y'+1) \\ &= 3x + 2y \end{aligned}$$

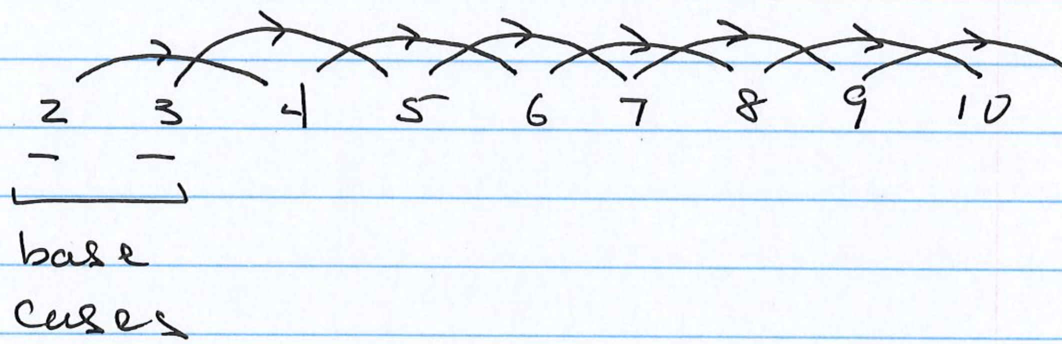
Result follows for $n \geq 2$ by 2nd PMI. ■

- why strong induction?

  since domino n was toppled by domino $n-2$, not domino $n-1$.

- Why two base cases?

dominos:

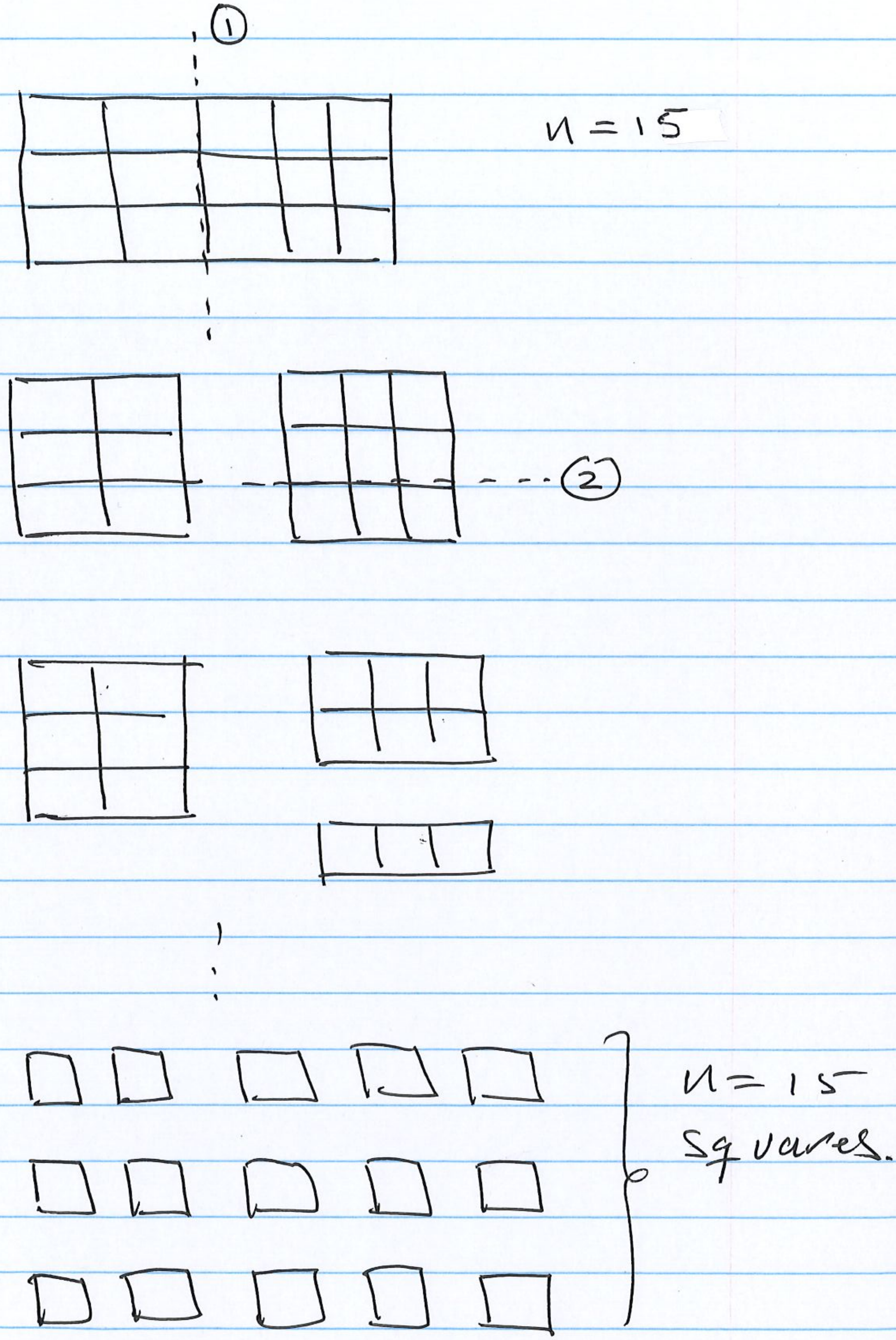


Exercise

Re-do this as weak induction form IIa or IIb with one base case.

split induction step into 2 cases: even & odd.

Hint: (5.2) # 10



5.3 Recursive Definitions

Read

- Recursively defined functions
- " " sets
- structural induction.

Recall: Fibonacci seq

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} & (n \geq 2) \end{cases}$$

Ex.

$$\forall n \geq 1 : \sum_{k=1}^{n} F_{2k-1} = F_{2n}$$