CSE 16 7-24-24

- Quiz 3 Tomorrow 7/25.

sections 2.5, 4.1, 4.3, 5.1

Ex. let $x \in \mathbb{R}$, $x \neq 1$. show

$P(n)$

$$\forall n \geq 1 : \boxed{\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}}$$

I. show $P(1)$ is true

$$x^0 = \frac{x^1 - 1}{x - 1}$$

i.e. $1 = 1$

which is true.

II. $\forall n \geq 1 : P(n) \to P(n+1)$

Let $n \geq 1$. Assume

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} \quad \left.\right\} \text{induction hypothesis}$$

we must show

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} \quad \left.\right\} \text{induction conclusion}$$

so

$$\sum_{k=0}^{n} x^k = \left(\sum_{k=0}^{n-1} x^k\right) + x^n$$

$$= \left(\frac{x^n - 1}{x - 1}\right) + x^n \quad \left.\right\} \text{by the induction hypothesis}$$

$$= \frac{x^n - 1 + x^n(x-1)}{x - 1}$$

$$= \frac{\cancel{x^n} - 1 + x^{n+1} - \cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1} - 1}{x - 1}$$

The result follows for all $n \geq 1$ by the P.M.I. ■

Rmks

- always state the ind. hyp. explicitely.
- always state the ind. conc. explicitely.
- always state at what point you use the induction hypothesis.

Ex. $\forall n \geq 1$ : $\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$ $P(n)$

I. $P(1)$ says $1 = \frac{1(1+1)}{2}$, i.e. $1=1$,

which is true

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

Let $n \geq 1$ be arbitrary. Assume

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ { ind. hyp.

we must show

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$ { ind. conc.

SO

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1)$$

$$= \left(\frac{n(n+1)}{2}\right) + (n+1) \quad \left\{ \begin{array}{l}\text{by the}\\ \text{ind. hyp.}\end{array}\right.$$

$$= (n+1)\left(\frac{n}{2} + \frac{2}{2}\right)$$

$$= \frac{(n+1)(n+2)}{2}.$$

Result follows for all $n \geq 1$ by the P.M.I. ∎

**Ex.** $\forall n \geq 1$ : $\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$

$P(n)$

Proof

I. P(1) says

$$1^2 = \frac{1\cdot(1+1)(2\cdot 1+1)}{6}$$

i.e. $1 = \frac{1\cdot 2\cdot 3}{6}$

i.e. $1 = 1$ ✓ true.

II. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$. Assume

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \Big\} \text{ ind. hyp.}$$

we must show

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6} \quad \Big\} \text{ ind. conc.}$$

so

$$\sum_{K=1}^{n+1} K^2 = \left(\sum_{K=1}^{n} K^2\right) + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \Big\{ \text{by the ind. hyp.}$$

$$= (n+1)\left(\frac{n(2n+1) + 6(n+1)}{6}\right)$$

$$= (n+1)\left(\frac{2n^2+n+6n+6}{6}\right)$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}.$$

Result follows by P.M.I. ∎

Exercise

Prove

$$\forall n \geq 1 : \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Sometimes $P(n)$ is false for some $n$, but true beyond some initial terms

$$\underbrace{P(1), P(2), \ldots, P(n_0-1)}_{\text{false}}, \underbrace{P(n_0), P(n_0+1) \ldots}_{\text{true}}$$

Base case: $n = n_0$.

To Prove: $\forall n \geq n_0 : P(n)$

I. show $P(n_0)$ is true

II. show $\forall n \geq n_0 : P(n) \rightarrow P(n+1)$

Ex. $\forall n \geq 4 : \boxed{5n+8 < n^2+4n+1}$ P(n)

note: $P(1)$, $P(2)$ and $P(3)$ are false

Proof

I. $P(4)$ says

$$5 \cdot 4 + 8 < 4^2 + 4 \cdot 4 + 1$$

i.e. $28 < 33$

which is true.

II. $\forall n \geq 4: P(n) \Rightarrow P(n+1)$

Let $n \geq 4$. Assume

$$5n+8 < n^2+4n+1 \quad \text{\{ ind. hyp.}$$

we must show

$$5(n+1)+8 < (n+1)^2+4(n+1)+1 \quad \text{\{ ind. conc.}$$

so

$$\begin{aligned}
5(n+1)+8 &= 5n+5+8 \\
&= (5n+8)+5 \\
&< (n^2+4n+1)+5 && \text{\{ by the ind. hyp.} \\
&= n^2+4n+6 \\
&< n^2+4n+6+2n && \text{\{ } n \geq 4 \Rightarrow 2n \geq 8 \Rightarrow 2n > 0
\end{aligned}$$

$$= n^2 + 6n + 6$$

$$= (n+1)^2 + 4(n+1) + 1 \quad \text{\{ by some algebra}$$

$\therefore \; 5(n+1) + 8 < (n+1)^2 + 4(n+1) + 1$ .

Result follows by PMI. ∎

Rmk. why add $2n$ ?

had : $n^2 + 4n + 6$

wanted : $(n+1)^2 + 4(n+1) + 1$

difference : $\left[(n+1)^2 + 4(n+1) + 1\right] - (n^2 + 4n + 6)$

$$= \underbrace{\cdots}_{\text{algebra}} = 2n$$

Ex. Let $A_1, A_2, A_3, \ldots$ be subsets of $U$. Then

P(n)

$$\forall n \geq 1 : \boxed{\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}}$$

Proof

I. $P(1)$ says $\overline{A_1} = \overline{A_1}$, which is true.

II. $\forall n \geq 1 : P(n) \to P(n+1)$

Let $n \geq 1$ be arbitrary.

Assume

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k} \quad \Big\{ \text{ind. hyp.}$$

we must show

$$\overline{\bigcup_{k=1}^{n+1} A_k} = \bigcap_{k=1}^{n+1} \overline{A_k} \quad \Big\{ \text{ind. conc.}$$

so

$$\overline{\bigcup_{k=1}^{n+1} A_k} = \overline{\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}}$$

$$= \left(\overline{\bigcup_{k=1}^{n} A_k}\right) \cap \overline{A_{n+1}}$$

$$= \left(\bigcap_{k=1}^{n} \overline{A_k}\right) \cap \overline{A_{n+1}} \quad \Big\{ \begin{matrix}\text{by the} \\ \text{ind. hyp.}\end{matrix}$$

$$= \bigcap_{k=1}^{n+1} \overline{A}_k$$

Result follows for all $n \geq 1$ by P.M.I. ∎

Exercises

- $\forall n \geq 1$ : $\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}$

Let $P_1, P_2, P_3, \ldots$ be Propositions. Then

- $\forall n \geq 1$ : $\neg \left( \bigvee_{k=1}^{n} P_k \right) = \bigwedge_{k=1}^{n} \neg P_k$
- $\forall n \geq 1$ : $\neg \left( \bigwedge_{k=1}^{n} P_k \right) = \bigvee_{k=1}^{n} \neg P_k$

## 5.2 Strong induction & the Well ordering Property

To prove the Principle of mathematical induction, we need:

### Well ordering Property of $\mathbb{Z}^+$

Any non-empty subset of $\mathbb{Z}^+$ contains a least element.

**Theorem** (P.M.I).
Given any $P : \mathbb{Z}^+ \to \{F, T\}$, the proposition

$$\big(P(1) \wedge \forall n\,(P(n) \to P(n+1))\big) \to \forall n\, P(n)$$

is true.

<u>Proof</u> (of PMI using WOP)

Assume $P(1)$ and $\forall n\,(P(n) \to P(n+1))$ are both true. We must show $\forall n\, P(n)$ is true.

Let $S = \{n \in \mathbb{Z}^+ \mid \neg P(n)\}$, i.e. $S$ is the set of integers "n" at which $P(n)$ is false.

It will be sufficient to show $S = \emptyset$, for then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Assume (to get a $\times$) that $S \neq \emptyset$. By the W.O.P. $S$ contains a least element, call it $m$. Thus $\boxed{m \in S}$ and $m-1 \notin S$.

Since $P(1)$ is true, $1 \notin S$, so $m \neq 1$, and hence $m-1 \in \mathbb{Z}^+$.

Since $\forall n\,(P(n) \to P(n+1))$ is true, we have for $n = m-1$, that

$$P(m-1) \to P(m).$$

since $m-1 \notin S$, we have $P(m-1)$ is true. Now we see both

$$P(m-1)$$

and

$$P(m-1) \to P(m)$$

are true, hence $P(m)$ is true. Therefore $\boxed{m \notin S}$.

This $\times$ shows our assumption was false, so $S = \phi$.

$\therefore \forall n P(n)$ is true. $\blacksquare$

# Variations on Induction

We can reparametrize the induction step.

I. $P(1)$

IIb. $\forall n > 1 : \underbrace{P(n-1)}_{\text{ind. hyp.}} \rightarrow \underbrace{P(n)}_{\text{ind. conc.}}$

Ex. $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.}$ $P(n)$

<u>Proof</u>

I. Same ✓

IIb. Let $n > 1$. Assume

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} \quad \text{\{ ind. hyp.}$$

we must show

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{ind. conc.}$$

so

$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k\right) + n$$

$$= \left(\frac{n(n-1)}{2}\right) + n \quad \text{by the ind. hyp.}$$

$$= \frac{n(n-1)+2n}{2}$$

$$= \frac{n^2-n+2n}{2}$$

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}.$$

Exercise

Re-do all previous examples using IIb instead of IIa.