cse 16 7-16-24

Proof of: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ ✓

let

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

$$\therefore S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

$$\therefore S + S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$$

$$\therefore 2S = n(n+1)$$

$$\therefore S = \frac{n(n+1)}{2}$$

∎

Another formula

$$\sum_{K=0}^{n} ar^{n} = \begin{cases} a(n+1) & \text{if } r=1 \\ a\left(\dfrac{r^{n+1}-1}{r-1}\right) & \text{if } r\neq 1 \end{cases}$$

<u>Proof of $r\neq 1$ case:</u>

let

$$rS = \quad\quad a\cancel{r} + a\cancel{r^2} + a\cancel{r^3} + \cdots + a\cancel{r^n} + ar^{n+1}$$

$$S = a + a\cancel{r} + a\cancel{r^2} + \cdots + a\cancel{r^n}$$

$$rS - S = ar^{n+1} - a$$

$$(r-1)S = a(r^{n+1}-1)$$

$$\therefore \quad S = a\left(\frac{r^{n+1}-1}{r-1}\right) \quad \blacksquare$$

Ex.

$$\sum_{k=5}^{10} 3\cdot 2^k = \sum_{k=0}^{10} 3\cdot 2^k - \sum_{k=0}^{4} 3\cdot 2^k$$

$$= 3\cdot\left(\frac{2^{11}-1}{2-1}\right) - 3\left(\frac{2^5-1}{2-1}\right)$$

$$= 3\left(2^{11}-2^5\right)$$

$$= 3\left(2048-32\right)$$

$$= 3\cdot 2016 = \boxed{6048}$$

Ex. Recall example

$$\begin{cases} a_n = 4a_{n-1} - 3a_{n-2} \\ a_0 = 1 \\ a_1 = 7 \end{cases}$$

earlier step

$$\underbrace{a_n - a_{n-1}}_{b_n} = 3\underbrace{(a_{n-1} - a_{n-2})}_{b_{n-1}}$$

let $b_n = a_n - a_{n-1}$. Then

$$\begin{cases} b_n = 3b_{n-1} \\ b_0 = 2 \end{cases} \xrightarrow{\text{soln.}} b_n = 2\cdot 3^n$$

$$\therefore \quad a_n - a_{n-1} = 2\cdot 3^n$$

$$\therefore \quad \begin{cases} a_n = a_{n-1} + 2\cdot 3^n \\ a_0 = 1 \end{cases}$$

so $a_1 = 1 + 2\cdot 3$

$$a_2 = 1 + 2\cdot 3 + 2\cdot 3^2$$

$$a_3 = 1 + 2\cdot 3 + 2\cdot 3^2 + 2\cdot 3^3$$

$$\vdots$$

$$a_n = 1 + 2\cdot 3 + 2\cdot 3^2 + \cdots + 2\cdot 3^n$$

$$= 1 - 2\cdot 3^0 + \left(2\cdot 3^0 + 2\cdot 3^1 + \cdots + 2\cdot 3^n\right)$$

$$= -1 + \sum_{k=0}^{n} 2\cdot 3^k$$

$$= -1 + \cancel{2}\left(\frac{3^{n+1}-1}{\cancel{3-1}}\right) = -2 + 3^{n+1}$$

$$\therefore \quad \boxed{a_n = -2 + 3^{n+1}}$$

(2.5) Cardinality of sets

Defn

Two sets A, B (not necessarily finite) are said to have the *same cardinality* iff there exists a bijection $f : A \to B$

notation: $|A| = |B|$

Ex. $|\mathbb{N}| = |\mathbb{Z}^+|$

why?

$$\begin{array}{ccc} \underline{\mathbb{N}} & \xrightarrow{f} & \underline{\mathbb{Z}^+} \\ 0 & \longrightarrow & 1 \\ 1 & \longrightarrow & 2 \\ 2 & \longrightarrow & 3 \\ 3 & \longrightarrow & 4 \\ \vdots & \vdots & \vdots \end{array}$$

$$f(n) = n+1$$

<u>Ex</u> let $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$

$O = \{n \in \mathbb{N} \mid n \text{ is odd}\}$

Then

- $|\mathbb{N}| = |E| \quad f(n) = 2n$
- $|\mathbb{N}| = |O| \quad f(n) = 2n+1$

<u>Ex</u> $|\mathbb{N}| = |\mathbb{Z}|$ ✓

$$\begin{array}{ccc} 0 & \overset{f}{\longmapsto} & 0 \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & -1 \\ 3 & \longmapsto & 2 \\ 4 & \longmapsto & -2 \\ \vdots & & \vdots \end{array}$$

$$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ even} \\ \frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

Exercise

Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Defn

A set S is called countable iff either

- S is finite ($|S| < \infty$)

or

- $|S| = |\mathbb{N}|$ (or $|\mathbb{Z}^+|, |\mathbb{Z}|, \ldots$)

In 2nd case we say S is countably infinite.

Informally: countably infinite means S forms an infinite list.

If S is not countable, it's called <u>uncountable</u>.

<u>Ex.</u> $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable

$$\begin{array}{ccccc}
11 & 12 & 13 & 14 & \cdots \\
21 & 22 & 23 & 24 & \cdots \\
31 & 32 & 33 & 34 & \cdots \\
41 & 42 & 43 & 44 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Exercise (see (2.5) #31 P. 177)

find an explicit formula for the bijection

$$f : \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$$

$$\begin{aligned} 1 &\mapsto 11 \\ 2 &\mapsto 21 \\ 3 &\mapsto 12 \\ 4 &\mapsto 31 \\ &\vdots \qquad \vdots \end{aligned}$$

Exercise

show that any subset of a countable set is countable. Hint: it is sufficient to show any subset of $\mathbb{Z}^+$ is countable.

Exercise

show that $\mathbb{Q}$ is countable.

Hint: find a bijection from $\mathbb{Q}^+$ to a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$.

Theorem

$\mathbb{R}$ is uncountable

Proof (contradiction)

Assume $\mathbb{R}$ is countable. Then so is the subset $S \subseteq \mathbb{R}$

$$S = (0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$$

Hence, S is the range of a sequence, i.e.

$$S = \{r_1, r_2, r_3, r_4, \ldots\ldots\}$$

Each element of S has a unique* decimal expansion.

$$\begin{array}{l} r_1 = .\,d_{11}\,d_{12}\,d_{13}\,\ldots\ldots \\ r_2 = .\,d_{21}\,d_{22}\,d_{23}\,\ldots\ldots \\ r_3 = .\,d_{31}\,d_{32}\,d_{33}\,\ldots\ldots \\ \vdots \qquad \vdots \quad \vdots \quad \vdots \end{array}$$

where each $d_{ij} \in \{0, 1, 2, \ldots, 9\}$

* If x has two decimal expansions choose the one with tail all 9's, not all 0's

$$.4999\ldots = \cancel{.5000}\ldots\ldots$$

Define $x \in S$ by the rule

$$x = .c_1 c_2 c_3 \ldots\ldots$$

where

$$c_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

observe $c_i \neq d_{ii}$ for all $i = 1, 2, 3, \ldots$

$\therefore$ $x \neq r_i$ for any $i = 1, 2, 3 \ldots$

Hence $\boxed{x \notin S}$. But clearly $0 < x \leq 1$, so $\boxed{x \in S}$. This contradiction shows our assumption was false, and $\mathbb{R}$ is uncountable.

∎

<u>Rmks</u>

- called the <u>cantor diagonal argument</u>.
- we've proved $|\mathbb{N}| \neq |\mathbb{R}|$.

Theorem

If $A_1, A_2, \ldots$ is a seq. of countable sets, then their union

$$S = \bigcup_{i=1}^{\infty} A_n$$

is also countable.

Exercise

Prove this. Hint: Similar to proof of $|\mathbb{Z}^+ \times \mathbb{Z}^+| = |\mathbb{Z}^+|$

<u>Theorem</u>

Let S be a set. Then no function

$$f : S \to \mathbb{P}(S)$$

is surjective.