CSE 16 7-11-24

2.4 Sequences & Summations

Defn

A *sequence* is a function whose domain is a subset of $\mathbb{Z}$, usually $\mathbb{N}$ or $\mathbb{Z}^+$.

Rmks

- write a_n instead of $a(n)$ for the image of n under a.
- a_n is called the n^{th} term of the sequence.

- Book writes $\{a_n\}$ for the sequence.

we will write

$$(a_n) = (a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, a_3, \dots)$$

- If the domain is finite we call it a <u>finite sequence</u>, otherwise its an <u>infinite sequence</u>.

<u>Ex.</u>

- $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$

- $(2^n)_{n=0}^{\infty} = (1, 2, 4, 8, 16, \dots)$

- $(k^2)_{k=3}^{6} = (9, 16, 25, 36)$

Defn

A Geometric Sequence is of the form

$$(a\cdot r^n)_{n=0}^{\infty} = (a, ar, ar^2, ar^3, \ldots\ldots)$$

a is the initial term

r is the common ratio

Ex. $(2\cdot 3^n) = (2, 6, 18, 54, \ldots)$

Defn

An Arithmetic Sequence is of the form

$$(a+nd)_{n=0}^{\infty} = (a,\, a+d,\, a+2d,\, a+3d,\, \ldots)$$

a is the initial term

d is the common difference

Ex.

$$(-1+4n) = (-1, 3, 7, 11, \ldots)$$

<u>Defn</u>

A <u>recurrence relation</u> for (a_n) is an equation that expresses a_n as a function of preceding terms: $a_{n-1}, a_{n-2}, \ldots, a_0$

<u>Ex</u> $a_n = a_{n-1} + 3$, $a_0 = 2$

$(a_n) = (2, 5, 8, 11, \ldots)$

<u>1st order</u>

<u>Ex.</u> $a_n = a_{n-1} - a_{n-2}$, $\underbrace{a_0 = 3,\ a_1 = 5}_{\text{initial terms}}$

$(a_n) = (3, 5, 2, -3, -5, -2, 3, \ldots)$

<u>2^{nd} order</u>, <u>Periodic</u>

<u>Ex.</u> Fibonacci Seq.

$F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$

$(F_n) = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$

Ex. Lucas seq.

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \ L_1 = 1$$

$$(L_n) = (2, 1, 3, 4, 7, 11, \ldots)$$

Defn

A solution to a recurrence *

$$* \quad a_n = F(a_{n-1}, \ldots)$$

is a closed formula for a_n (a function of n only) $a_n = f(n)$, that when substituted into LHS and RHS of * yields an identity for all n.

Ex. $\begin{cases} a_0 = 1 \\ a_n = n \cdot a_{n-1} \end{cases}$

Solution: $a_n = n!$

$a_0 = 1$

$a_1 = 1 \cdot 1$

$a_2 = 2 \cdot 1 \cdot 1$

$a_3 = 3 \cdot 2 \cdot 1 \cdot 1$

$a_4 = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1$

$a_5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1$

$\vdots$

Ex.

$$\begin{cases} b_0 = 1 \\ b_n = 3 \cdot b_{n-1} \end{cases}$$

solution: $b_n = 3^n$

$$b_0 = 1$$

$$b_1 = 3 \cdot 1$$

$$b_2 = 3 \cdot 3 \cdot 1$$

$$b_3 = 3 \cdot 3 \cdot 3 \cdot 1$$

$$b_4 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 1$$

Ex $c_n = 5c_{n-1} - 6c_{n-2}$

check
- $c_n = 2^n$ $\boxed{\text{yes}}$
- $c_n = 3^n$ $\boxed{\text{yes}}$
- $c_n = 5^n$ $\boxed{\text{no}}$

check $c_n = 2^n$

$$\begin{aligned}
RHS &= 5c_{n-1} - 6c_{n-2} \\
&= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} \\
&= 5 \cdot 2^{n-1} - 3 \cdot 2 \cdot 2^{n-2} \\
&= 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} \\
&= (5-3) \cdot 2^{n-1} \\
&= 2 \cdot 2^{n-1} \\
&= 2^n = c_n = LHS
\end{aligned}$$

check $c_n = 5^n$ X

$$RHS = 5 \cdot c_{n-1} - 6 c_{n-2}$$

$$= 5 \cdot 5^{n-1} - 6 \cdot 5^{n-2}$$

$$= 5^{n-2}(25-6) = 19 \cdot 5^{n-2}$$

$$LHS = c_n = 5^n \neq 19 \cdot 5^{n-2}$$

__Exercise__ check $c_n = 3^n$ __is__ a solution

__Exercise__

- Fibonacci: $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$
- Lucas: $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

<u>Hint</u> let $\phi = \frac{1+\sqrt{5}}{2}$, $\bar{\phi} = \frac{1-\sqrt{5}}{2}$

show that ϕ (the golden ratio) and $\bar{\phi}$ are roots of the quadratic

$$x^2 - x - 1 = 0$$

$$x^2 = x + 1$$

i.e. $\phi^2 = \phi + 1$, $\bar{\phi}^2 = \bar{\phi} + 1$

Ex. find a recurrence for

$a_0 \quad a_1 \quad a_2 \quad \dots \quad a_{n-2} \quad a_{n-1} \quad a_n$

1, 7, 25, 79, 241, 727, 2185, 6559, ...

diff: 6 18 54 162 486 1458 4374

ratio: 3 3 3 3 3 3

so $(a_n - a_{n-1}) = 3(a_{n-1} - a_{n-2})$

$$\therefore \quad \begin{cases} a_n = 4a_{n-1} - 3a_{n-2} \\ a_0 = 1 \quad \checkmark \\ a_1 = 7 \quad \checkmark \end{cases}$$

<u>check</u> solution is

$$\boxed{a_n = 3^{n+1} - 2}$$

$$\begin{aligned}
\text{RHS} &= 4a_{n-1} - 3a_{n-2} \\
&= 4(3^n - 2) - 3(3^{n-1} - 2) \\
&= 4\cdot 3^n - 8 - 3^n + 6 \\
&= 3\cdot 3^n - 2 \\
&= 3^{n+1} - 2 \\
&= a_n = \text{LHS}.
\end{aligned}$$

Defn

A summation (or series) is the sum of a sequence, which may be finite or infinite.

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

Ex. $\displaystyle\sum_{j=0}^{6} 2^j = 1+2+4+8+16+32+64$

$$= \boxed{127}$$

Ex $\displaystyle\sum_{k\in\{2,4,6\}} k^2 = 2^2+4^2+6^2$

Special formulas

(1) $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(2) $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(3) $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(4) $$\sum_{k=0}^{n} ar^k = \begin{cases} a(n+1), & \text{if } r=1 \\ a\left(\frac{r^{n+1}-1}{r-1}\right) & \text{if } r\neq 1 \end{cases}$$