CSE 16 7-10-24

Defn

let $f: A \to B$ and $S \subseteq A$. The **image** of S under f is the set

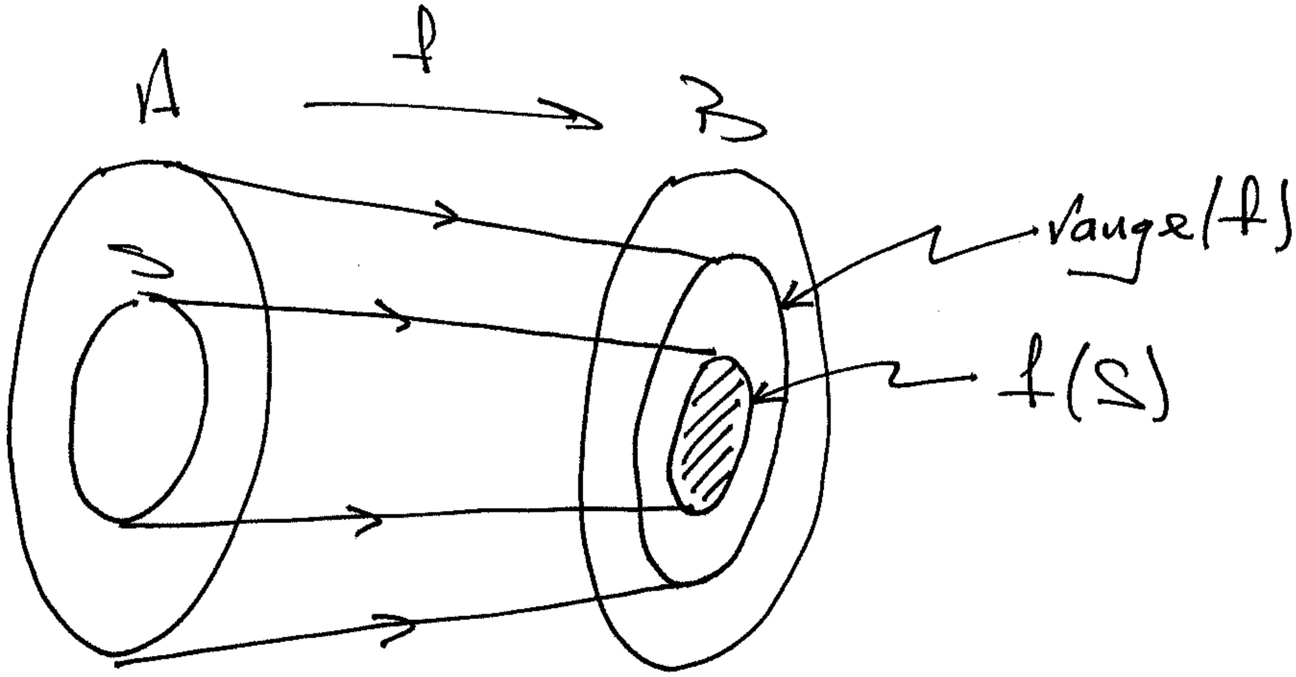
$$f(S) = \{ y \in B \mid \exists x \in S : y = f(x) \}$$

$$= \{ f(x) \mid x \in S \}$$

note:

$$f(S) \subseteq f(A) = \text{range}(f) \subseteq B$$

Picture:



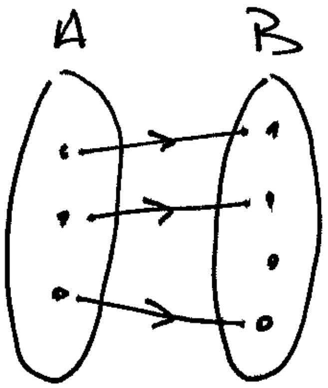
Defn

A function $f: A \to B$ is called injective (one-to-one) iff

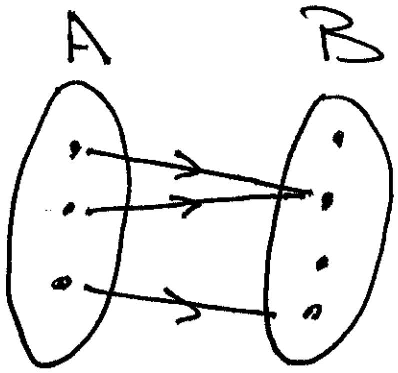
$$\forall x_1, x_2 \in A : f(x_1) = f(x_2) \rightarrow x_1 = x_2$$

equivalently

$$\forall x_1, x_2 \in A : x_1 \neq x_2 \longrightarrow f(x_1) \neq f(x_2)$$



injective



not injective

Ex $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$

not injective ✓

Proof:

let $x_1=1$, $x_2=-1$. Then $f(x_1)=1^2=1$ and $f(x_2)=(-1)^2=1=f(x_1)$. Thus

$$\exists x_1,x_2\in\mathbb{R}: x_1\neq x_2 \text{ and } f(x_1)=f(x_2)$$

$$\therefore \neg\forall x_1,x_2\in\mathbb{R}: x_1\neq x_2 \to f(x_1)\neq f(x_2)$$

∎

Ex $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$

g is injective ✓

Proof

Let $x_1, x_2 \in \mathbb{R}$ be chosen arbitrarily.

Assume $g(x_1) = g(x_2)$. We must show $x_1 = x_2$. We have

$$2x_1 + 3 = 2x_2 + 3$$

$$\therefore \quad 2x_1 = 2x_2$$

$$\therefore \quad x_1 = x_2 \, .$$

▨

Defn

we say $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing iff

$$\forall x_1, x_2 : \; x_1 < x_2 \to f(x_1) < f(x_2)$$

and strictly decreasing iff

$$\forall x_1, x_2 : \; x_1 < x_2 \to f(x_1) > f(x_2).$$

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is either strictly increasing or strictly decreasing, then f is injective.

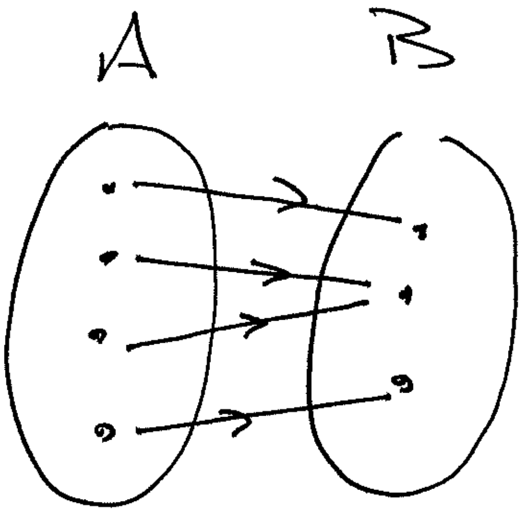
Defn

$f : A \to B$ is called surjective (or onto) iff

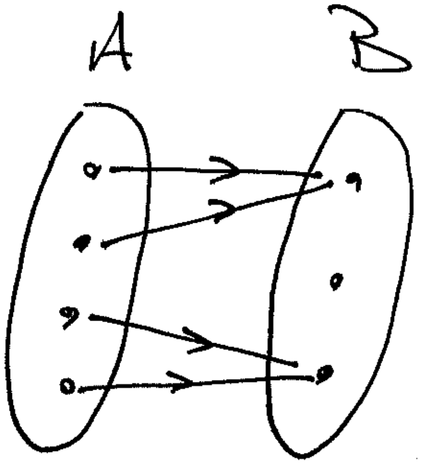
$$\text{range}(f) = B$$

equivalently

$$\forall y \in B, \exists x \in A : y = f(x)$$



surjective



not surjective

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

not surjective ✓

Proof

Let $y = -1$. Then $\forall x: x^2 \neq -1$.

$\therefore \exists y \in \mathbb{R}, \forall x \in \mathbb{R}: f(x) \neq y$

$\therefore \neg \forall y \in \mathbb{R}, \exists x \in \mathbb{R}: y = f(x)$.

▨

Ex. $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x + 3$.

surjective ✓

Proof

Let $y \in \mathbb{R}$ be arbitrary.

Let $x = \frac{y-3}{2}$. Then

$$
\begin{aligned}
g(x) &= g\left(\frac{y-3}{2}\right) \\
&= 2\left(\frac{y-3}{2}\right) + 3 \\
&= (y-3) + 3 \\
&= y.
\end{aligned}
$$

$\therefore \ \forall y \in \mathbb{R}, \exists x \in \mathbb{R} \,!\, \ y = g(x).$

$\blacksquare$

Defn

A function $f: A \to B$ is called **bijective** iff it is both injective and surjective.

Ex. $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$ is a bijection.

Defn

The **composition** of $f: A \to B$ and $g: B \to C$ is the function

$$g \circ f : A \to C$$

given by

$$g \circ f(x) = g(f(x))$$

for any $x \in A$.

Ex. $A = B = C = \mathbb{R}$ and

$$f(x) = x^2$$

$$g(x) = 2x+3$$

Then

$$g \circ f(x) = g(f(x)) = g(x^2) = 2x^2+3$$

and

$$f \circ g(x) = f(g(x)) = f(2x+3) = (2x+3)^2$$

note: $f \circ g \neq g \circ f$

Theorem

Let $f: A \to B$, $g: B \to C$. Then

(a) if f and g are both injective, then so is $g \circ f : A \to C$

(b) if f and g are both surjective, then so is $g \circ f$.

(c) if f and g are both bijective, then so is $g \circ f$.

Proof of (a)

Assume f and g are injective.

Let $x_1, x_2 \in A$ be arbitrary.

suppose that $g \circ f(x_1) = g \circ f(x_2)$.

we must show that $x_1 = x_2$.

we have

$$g(f(x_1)) = g(f(x_2))$$

$\therefore \quad f(x_1) = f(x_2)$ since g is inj.

$\therefore \quad x_1 = x_2$ since f is inj.

∎

Exercise: Prove (b)

Exercise: Prove that composition is associative. i.e. show that if $f: A \to B$, $g: B \to C$, $h: C \to D$, then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Invertibility

Define for sets A, B

$$i_A : A \to A \quad \text{by} \quad i_A(x) = x$$

$$i_B : B \to B \quad \text{by} \quad i_B(y) = y$$

Note that for any $f: A \to B$,

$$f \circ i_A = f = i_B \circ f$$

$$\begin{tikzcd}
A \arrow[r, "i_A"] & A \arrow[r, "f"] & B \arrow[r, "i_B"] & B \\
x \arrow[r, mapsto] & x \arrow[r, mapsto] & y \arrow[r] & y
\end{tikzcd}$$

i_A and i_B are called <u>identity functions</u>.

<u>Defn</u>

A function $f: A \to B$ is called <u>invertible</u> iff there exists $g: B \to A$ such that

$$f \circ g = i_B \quad \text{and} \quad g \circ f = i_A$$

function $g: B \to A$ is called an inverse function for $f: A \to B$.

Theorem

If f is invertible, then its inverse g is unique.

Proof

suppose there exist

$$g_1 : B \to A \text{ and } g_2 : B \to A$$

satisfying

- $\boxed{f \circ g_1 = i_B}$ and $g_1 \circ f = i_A$

and

- $f \circ g_2 = i_B$ and $\boxed{g_2 \circ f = i_A}$

Then

$$
\begin{aligned}
g_1 &= i_A \circ g_1 \\
&= (g_2 \circ f) \circ g_1 \\
&= g_2 \circ (f \circ g_1) \\
&= g_2 \circ i_B \\
&= g_2 .
\end{aligned}
$$

▨.

We may now speak of *the* inverse fcn. to f and write

$$f^{-1} : B \to A$$

for g.

Theorem

$f : A \to B$ is invertible $\overset{\Longrightarrow}{\underset{\Longleftarrow}{\text{iff}}}$ f is bijective.

Proof

($\Longrightarrow$) Assume f is invertible, so $f^{-1} : B \to A$ exists.

- f is injective

  Proof: let $x_1, x_2 \in A$ and assume $f(x_1) = f(x_2)$.

$$\therefore\ f^{-1}(f(x_1)) = f^{-1}(f(x_2))$$

$$\therefore\ f^{-1} \circ f(x_1) = f^{-1} \circ f(x_2)$$

$$\therefore\ i_A(x_1) = i_A(x_2)$$

$$\therefore \quad x_1 = x_2 \qquad \square$$

- <u>f is surjective</u>

Let $y \in B$ be arbitrary. Then set $x = f^{-1}(y)$.

$$\begin{aligned} \therefore \quad f(x) &= f\left(f^{-1}(y)\right) \\ &= f \circ f^{-1}(y) \\ &= 1_B(y) \\ &= y \qquad \square \end{aligned}$$

We've shown $f: A \to B$ is bijective.

▨

$(\Leftarrow)$ Assume f is bijective.

Define $g: B \to A$ by:

for each $y \in B$

$g(y) =$ the unique $x \in A$
such that $y = f(x)$

<u>Note</u>: Such an x exists since f is surjective, and x is unique since f is injective.

Then necessarily

$f(g(y)) = y$ for all $y \in B$

and

$g(f(x)) = x$ for all $x \in A$

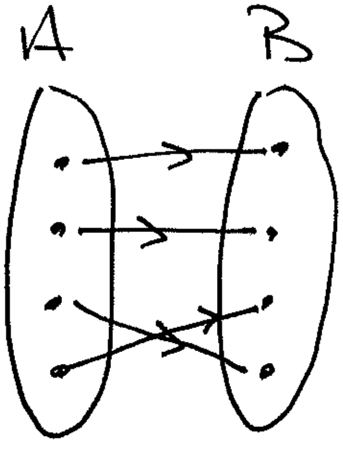
Thus

$$f \circ g = i_B \quad \text{and} \quad g \circ f = i_A$$

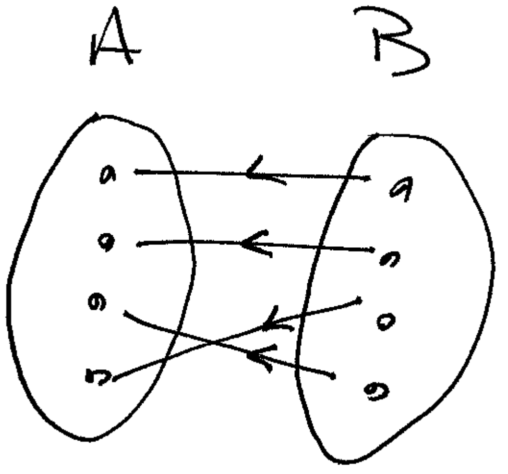
$\therefore$ $g = f^{-1}$, and f is invertible.

▨

Picture



f



f^{-1}

Ex. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$

is invertible and

$$g^{-1}(y) = \frac{y-3}{2}$$

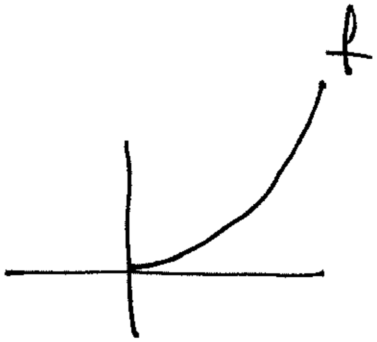
Ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is

not invertible.

Ex. $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$

is invertible and

$$f^{-1}(x) = \sqrt{x}$$



also $h : \mathbb{R}^- \to \mathbb{R}^+$, $h(x) = x^2$

$$h^{-1}(x) = -\sqrt{x}$$