Ex. 'all horses in California are blue'

- let $\mathcal{U} = \{\text{horses in CA}\}$, and let $B(x) = $ 'x is blue'

$$\forall x \; B(x)$$

- let $\mathcal{U} = \{\text{horses}\}$, $C(x) = $ 'x lives in CA'

$$\forall x \; (C(x) \rightarrow B(x))$$

- let $\mathcal{U} = \{\text{living things}\}$, $H(x) = $ 'x is a horse'

$$\forall x \; (H(x) \wedge C(x) \rightarrow B(x))$$

This is wrong:

$$\forall x\,(H(x) \wedge C(x) \wedge B(x))$$

1.5 Nested Quantifiers

$\equiv$ for 1^{st} order logic.

<u>Defn</u>

Let A, B be expressions with Predicates and Quantifiers, and in which all Variables are bound. Then $A \equiv B$ iff A and B have the same truth value for all Propositional functions and all Universes.

Ex $\neg \forall_x P(x) \equiv \exists_x \neg P(x)$ ✓

Ex $\neg \forall_x P(x) \not\equiv \forall_x \neg P(x)$ ✓

why? let $P(x) =$ '$x > 0$', $U = \mathbb{R}$.

Then : LHS is true

RHS is false

Let $P(x,y)$ where $U_x = U_y = U$.

Then

(1) $\forall_x \forall_y P(x,y) \equiv \forall_y \forall_x P(x,y)$

(2) $\exists_x \exists_y P(x,y) \equiv \exists_y \exists_x P(x,y)$

(3) $\forall_x \exists_y P(x,y) \not\equiv \exists_y \forall_x P(x,y)$

Illustrate (1) with $U = \{1, 2\}$

$$\forall x \forall y P(x,y) \equiv \forall y P(1,y) \wedge \forall y P(2,y)$$

$$\equiv \big(P(1,1) \wedge P(1,2)\big) \wedge \big(P(2,1) \wedge P(2,2)\big)$$

and

$$\forall y \forall x P(x,y) \equiv \forall x P(x,1) \wedge \forall x P(x,2)$$

$$\equiv \big(P(1,1) \wedge P(2,1)\big) \wedge \big(P(1,2) \wedge P(2,2)\big)$$

By assoc. & comm. we have

$$LHS \equiv RHS$$

Proof of (3)

Let $P(x,y) =$ '$x < y$', $U = \mathbb{R}$.

Then

$\forall_x \exists_y P(x,y)$ is true

$\exists_y \forall_x P(x,y)$ is false

More Translation

Let $L(x,y) =$ 'x loves y', where

$U_x = U_y = \{\text{people}\}$

Ex.

(1) 'everybody loves somebody'

$$\forall x \, \exists y \, L(x,y)$$

(2) 'somebody loves everybody'

$$\exists x \, \forall y \, L(x,y)$$

(3) 'nobody loves everybody'

$$\neg \exists x \, \forall y \, L(x,y)$$

$$\equiv \forall x \, \neg \forall y \, L(x,y)$$

$$\equiv \forall x \, \exists y \, \neg L(x,y)$$

(4) 'there is exactly one person whom everyone loves'

$$\exists y \left(\forall x\, L(x,y) \wedge \forall z \left(z \neq y \rightarrow \exists w\, \neg L(w,z) \right) \right)$$

(5) 'there is a person who loves everyone except himself'

$$\exists x \left(\neg L(x,x) \wedge \forall y \left(y \neq x \rightarrow L(x,y) \right) \right)$$

Ex.

let $U = \{\text{people}\}$, and

$E(x) =$ 'x is the King of England'

$F(x) =$ 'x is the King of France'.

Translate: 'no King of England is not the King of France'.

$$\forall x \, (E(x) \to F(x))$$

$$\equiv \forall x \, (\neg E(x) \vee F(x))$$

$$\equiv \forall x \, (\neg E(x) \vee \neg\neg F(x))$$

$$\equiv \forall x \, \neg (E(x) \wedge \neg F(x))$$

$$\equiv \neg \exists x \, (E(x) \wedge \neg F(x))$$

Scope of Quantifiers

(1) $\forall x\,(P(x) \to Q(x)) \not\equiv \forall x\,P(x) \to \forall x\,Q(x)$

(2) $\exists x\,(P(x) \to Q(x)) \not\equiv \exists x\,P(x) \to \exists x\,Q(x)$

(3) $\exists x\,(P(x) \wedge Q(x)) \not\equiv \exists x\,P(x) \wedge \exists x\,Q(x)$

- (1) and (2) are $\not\equiv$ since
  $U = \{\text{people}\}$
  $P(x) =$ 'x has green eyes'
  $Q(x) =$ 'x is 50 ft. tall'
- (3) is $\not\equiv$ since: $U = \mathbb{R}$,
  $P(x) =$ '$x > 0$', $Q(x) =$ '$x < 0$'

<u>Exercise</u> investigate

(4) $\forall x\,(P(x) \vee Q(x))$? $\forall x P(x) \vee \forall x Q(x)$

(5) $\exists x\,(P(x) \vee Q(x))$? $\exists x P(x) \vee \exists x Q(x)$

(6) $\forall x\,(P(x) \wedge Q(x))$? $\forall x P(x) \wedge \forall x Q(x)$

<u>Hint</u>: check $U = \{1, 2\}$

From 1.1 & 1.2 on Puzzles

self referencing statements or groups of ~~statements~~ can cause logical dependencies.

Ex.

$P =$ 'this statement is false'

$P = 1 \Rightarrow P = 0$

$P = 0 \Rightarrow P = 1$

$\therefore$ P is not proposition.

Ex.

p = 'stmt. q is false' $= \neg q = \neg\neg p$

q = 'stmt. p is false' $= \neg p = \neg\neg q$

p	q	
0	0	$\div$
0	1	Possible
1	0	Possible
1	1	$\div$

Ex.

① 'exactly 1 stmt in this list is false'
② ' " 2 " " " " " false'
③ ' " 3 " " " " " false'

①	②	③	
0	0	0	✗
0	0	1	✗
0	1	0	Possible
0	1	1	✗
1	0	0	✗
1	0	1	✗
1	1	0	✗
1	1	1	✗

Ex. 1.2 #24

we have 3 persons

A, B, C } one Knight (T), one Knave (F), one Spy (T or F)

A says: 'C is the Knave'

B says: 'A is the Knight'

C says: 'I am the spy'

who is who?

A	B	C
T	F	S
T	S	F
F	T	S
F	S	T
S	T	F
S	F	T