CSE 16 6-26-24

Ex. $p \to q \equiv \neg p \vee q$ ✓

p	q	$\neg p$	$\neg p \vee q$	$p \to q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	0	1	1	1

tautology

Exercise: Prove all equivalences on tables 6, 7, 8 in sec. 1.3

Proof of 1st DeMorgan

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

same

Algebraic Proofs

<u>Ex.</u> show $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

<u>Proof</u>

$$(p \vee q) \rightarrow r \equiv \neg(p \vee q) \vee r \qquad \text{T7 \#1}$$

$$\equiv (\neg p \wedge \neg q) \vee r \qquad \text{DeMorgan}$$

$$\equiv (\neg p \vee r) \wedge (\neg q \vee r) \qquad \text{comm. \& assoc.}$$

$$\equiv (p \rightarrow r) \wedge (q \rightarrow r) \qquad \text{T7 \#1}$$

■

<u>Ex.</u> show $(p \wedge q) \rightarrow (p \rightarrow q)$ is a tautology

<u>Proof</u>

$$(p \wedge q) \rightarrow (p \rightarrow q) \equiv \neg(p \wedge q) \vee (\neg p \vee q) \qquad \text{T7 \#1}$$

$$\equiv (\neg p \vee \neg q) \vee (\neg p \vee q) \qquad \text{DeMorgan}$$

$$\equiv (\neg p \vee \neg p) \vee (\neg q \vee q) \qquad \text{Comm, Assoc}$$

$$\equiv \neg p \vee T \qquad \text{Idem., Neg.}$$

$$\equiv T \qquad \text{Domination}$$

■

Exercise

show $\neg(p \Rightarrow q) \rightarrow p$ is a tautology.

Exercise

Determine whether

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

is satisfiable.

Note:

$$\equiv \quad (p \Rightarrow r) \wedge (r \Rightarrow q) \wedge (q \Rightarrow p)$$

1.4 Predicates & Quantifiers

Defn

A <u>Propositional Function</u> (or a <u>Predicate</u>) is a statement $P(x)$ that becomes a proposition when a value for x is specified.

Ex $P(x) =$ '$x < 7$' $\qquad \mathcal{U} = \mathbb{R}$

$P(5)$ is true : $5 < 7$

$P(8)$ is false : $8 < 7$

Ex. $P(x) =$ 'x is a sunny day'

$\mathcal{U} = \{\text{days}\}$

Ex. $Q(x,y) = 'x+y=10' : \mathcal{U}_x = \mathcal{U}_y = \mathbb{R}$

The Universe of Discourse or Domain of a Propositional function is the set of possible values for its variable(s). Write: $\mathcal{U}, \mathcal{U}_x, \mathcal{U}_y$

How to get a Proposition from a Proposition?

- one way: Substitute $x \in \mathcal{U}$ into $P(x)$
- another way: Quantification
  - $\forall$: universal
  - $\exists$: existential

Defn

The Universal Quantification of $P(x)$ is the proposition

'$P(x)$ is true for all $x \in \mathcal{U}$'

notation: $\forall x\ P(x)$

read: 'for all x, $P(x)$'

Defn

The Existential Quantification of $P(x)$ is the proposition

'$P(x)$ is true for at least one $x \in \mathcal{U}$'

<u>notation</u>: $\exists x\, P(x)$

<u>read</u>: 'there exists an x, such that $P(x)$'

<u>Ex.</u> $P(x) =$ ' $x^2 \geq 0$ ' , $U = \mathbb{R}$

$\forall x\, P(x)$ is 'the square of any real number is non-negative.'

$\exists x\, P(x)$ is 'the square of some real number is non-neg.'

Ex: $Q(x) = $ '$x < 50$' $\quad U = \mathbb{R}$

$\forall x\, Q(x)$ false

$\exists x\, Q(x)$ true

Ex. $R(x) = $ '$2x = 3$' $\quad U = \mathbb{R}$

$\forall x\, R(x)$ false

$\exists x\, R(x)$ true

Question: what if $U = \mathbb{Z}$

$\uparrow$

integers

consider the finite universe

$$U = \{1, 2\}$$

Then

$$\forall x\, P(x) \equiv P(1) \wedge P(2)$$

$$\exists x\, P(x) \equiv P(1) \vee P(2)$$

so

$$\neg \forall x\, P(x) \equiv \neg\left(P(1) \wedge P(2)\right)$$

$$\equiv \neg P(1) \vee \neg P(2)$$

$$\equiv \exists x\, \neg P(x)$$

$$\therefore \boxed{\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)}$$

and

$$\neg \exists_x P(x) \equiv \neg \left(P(1) \vee P(2) \right)$$

$$\equiv \neg P(1) \wedge \neg P(2)$$

$$\equiv \forall_x \neg P(x)$$

$$\therefore \boxed{\neg \exists_x P(x) \equiv \forall_x \neg P(x)}$$

also hold for any Prop. fcn.

on any universe.

	True when	False when
$\forall x\, P(x)$	$P(x)$ is true for all $x \in U$	$P(x)$ is false for some $x \in U$
$\exists x\, P(x)$	$P(x)$ is true for some $x \in U$	$P(x)$ is false for all $x \in U$

<u>Defn</u>

a variable x is called

- <u>Bound</u> if it has been either quantified or substituted-for
- <u>free</u> if it is not bound

Ex.

	x	y	z	Proposition ?
$P(x,y) \vee Q(z)$	f	f	f	no
$\forall x\, P(x,y) \vee Q(z)$	b	f	f	no
$\exists x\, \forall z\, P(x,y) \vee Q(z)$	b	f	b	no
$\exists z\, P(6,8) \vee Q(z)$	b	b	b	yes