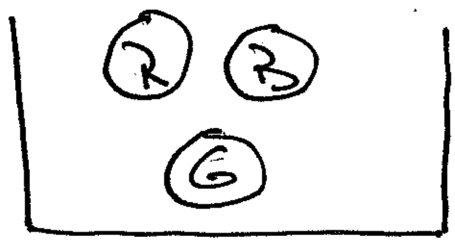
CSE 16 8-9-23

Defn

A <u>k-combination</u> of a set S <u>with repetition</u> is an unordered arrangement of the elements of S, where repetition is allowed. Also called a <u>k-multiset</u> of S.

Ex.

An urn contains 3 balls: Red, Blue, Green. Draw 4 balls with replacement. How many color combinations are possible.



RRRR, BBBB, GGGG
RRRB, RRRG,
GGGR, GGGB
BBBR, BBBG,
RRBB, RRGG, BBGG
RRBG, BBRG, GGRB

15 combinations

These are the 4-comb. from the 3-set $\{R, B, G\}$.

<u>Note</u>: each combination corresponds to a unique bit string on alphabet $\{*, 1\}$

RRBG $\longleftrightarrow$ $**|*|*$

RBBB $\longleftrightarrow$ $*|***|$

BGGG $\longleftrightarrow$ $|*|***$

RRGG $\longleftrightarrow$ $**||**$

BBBG $\longleftrightarrow$ $|***|*$

each string is of len. 6, and contains exactly 4 $*$'s and 2 $|$'s. The 2 $|$'s divide the string into 3 cells, representing colors R, B, G respectively. The # of $*$'s in each cell represents the # of times the corresponding color is selected.

This mapping

$$\{\text{color combinations}\} \to \{\text{bit strings}\}$$

is bijective.

Thus

$$(\#\text{color comb.s})$$

$$= (\#\text{strings})$$

$$= \binom{6}{4} = \binom{6}{2} = \frac{6!}{4!\,2!} = \boxed{15}$$

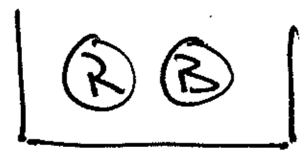
<u>Theorem</u>

Let $n, k \in \mathbb{N}$. The # of k-combinations from an n-set, with repetition, is

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

Exercise: prove this.

Ex. $n = 2,\ k = 3$



$$\binom{2+3-1}{3} = \binom{4}{3} = 4$$

$RRR \longrightarrow ***|$

$RRB \longrightarrow **|*$

$RBB \longrightarrow *|**$

$BBB \longrightarrow |***$

Ex. determine the sum

$$\sum_{k=1}^{10} \sum_{j=1}^{k} \sum_{i=1}^{j} 1$$

Each term corresponds to a unique triple (i, j, k) s.t. $1 \le i \le j \le k \le 10$

$$\text{same sum} = \sum_{1 \le i \le j \le k \le 10} 1$$

These triples are the 3-combinations from set $\{1, 2, 3, \ldots, 10\}$, with repetition.

$$\therefore \text{sum} = \binom{10+3-1}{3} = \binom{12}{3} = \boxed{220}$$

contrasting sum:

$$\sum_{k=3}^{10} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} 1 = \sum_{1 \le i < j < k \le 10} 1$$

This is # triples (i,j,k) s.t. $1 \le i < j < k \le 10$. These are 3-comb. from $\{1, 2, ..., 10\}$ <u>without</u> repetition.

$$\text{sum} = \binom{10}{3} = \boxed{120}$$

Ex.

How many solutions to

$$x_1 + x_2 + x_3 = 11$$

are there with $x_i \in \mathbb{N}$ $(1 \le i \le 3)$

Each solution (x_1, x_2, x_3) corresponds to an 11-combination from $\{1, 2, 3\}$ with repetition.

$x_1 = \#\,1\text{'s selected}$

$x_2 = \#\,2\text{'s}$ "

$x_3 = \#\,3\text{'s}$ "

so

$$\#\text{solns} = \binom{3+11-1}{11} = \binom{13}{11} = \boxed{78}$$

Summary

The # of ways to arrange k-elements from an n-set is

ordered?	repetition? no	repetition? yes
no (combination)	$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$	$\binom{n+k-1}{k}$
yes (permutation)	$P(n,k) = \frac{n!}{(n-k)!}$	n^k

Permutations with some objects indistinguishable

Ex.
how many anag-ams of "EVERGREEN"

1. EEEE
2. G
3. N
4. RR
5. V

let $m = \#$ anag-ams

Temporarily mark E's & R's so as to distinguish: $\{E_1, E_2, E_3, E_4\}$, $\{R_1, R_2\}$

Now count # Permutations of new Symbols, of which there are 9.

- choose an anagram of 'EVERGREEN'

  #ways $= m$

- choose a permutation of $\{E_1, E_2, E_3, E_4\}$

  #ways $= 4!$

- choose a permutation of $\{R_1, R_2\}$

  #ways $= 2!$

Thus, by product rule

$$9! = m \cdot 4! \cdot 2!$$

$$\therefore \quad m = \frac{9!}{4! \cdot 2!} = \boxed{7560}$$

Theorem

The # of permutations of n objects, where there are n_i indistinguishable objects of type i $(1 \leq i \leq k)$ and $n_1 + n_2 + \cdots + n_k = n$, is

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

In last example $k=5$, $n=9$

$n_1 = 4$, $n_2 = n_3 = n_5 = 1$, $n_4 = 2$.

Exercise: Prove this.

Ex.

find the number of ternary strings (alphabet $= \{0,1,2\}$) of length 20 containing exactly 3 0's, 12 1's, and 5 2's.

count anagrams of $0^3 1^{12} 2^5$ where

$0^3 = 000$
$1^{12} = 111111111111$
$2^5 = 22222$

By theorem

$$\text{ans} = \frac{20!}{3!\,12!\,5!} = \boxed{7{,}054{,}320}$$

Recall we can do this with binomial coeffs:

$$\text{ans} = \binom{20}{3}\binom{17}{12} = \frac{20!}{3!\,\cancel{17!}}\cdot\frac{\cancel{17!}}{12!\,5!}$$

choose 0 pos. choose 1 pos.

$$= \frac{20!}{3!\,12!\,5!} = \boxed{7{,}054{,}320}$$

<u>Defn</u>

Let $n = n_1 + n_2 + \cdots + n_k$ where $n_i \in \mathbb{N}$ $(1 \le i \le k)$. The <u>multinomial coefficient</u>

$$\binom{n}{n_1, n_2, \ldots, n_k}$$

is the number of permutations of the n-element multiset

$$\{x_1^{n_1}, x_2^{n_2}, \ldots, x_k^{n_k}\}$$

i.e. anagrams of

$$\overbrace{x_1 \cdots x_1}^{n_1}\ \overbrace{x_2 \cdots x_2}^{n_2} \cdots \overbrace{x_k \cdots x_k}^{n_k}.$$

By the last theorem

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

Exercise

- $\binom{n}{r,\, n-r} = \binom{n}{r}$

- $\binom{n}{a,\, b,\, n-a-b} = \binom{n}{a} \cdot \binom{n-a}{b}$

- etc...

Multinomial Theorem

let $x_1, \dots, x_k \in \mathbb{R}$. Then

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{n_1 + \dots + n_k = n} \binom{n}{n_1, \dots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

where sum is over all k-tuples $(n_1, \dots, n_k)$ s.t. $n_1 + \dots + n_k = n$.

Exercise: Prove this.

7.1 Discrete Probability

Defns

- an experiment is a procedure that yields one of a set of possible outcomes.
- The set of outcomes is called the sample space : S
- An event is a subset of the sample space: $E \subseteq S$.

- Let S be a finite sample space and $E \subseteq S$ an event. The probability of E is

$$P(E) = \frac{|E|}{|S|}$$

<u>Observe</u>:

- for all $E \subseteq S$: $0 \leq P(E) \leq 1$
- $P(\emptyset) = 0$
- $P(S) = 1$
- if $E_1, \ldots, E_k \subseteq S$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, then

$$P(E_1 \cup \cdots \cup E_k) = P(E_1) + \cdots + P(E_k).$$