CSE 16 8-4-23

Continuing example ...

after collecting like terms, have

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

<u>Binomial Theorem</u>

for $n \in \mathbb{N}$, $x, y \in \mathbb{R}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

<u>Proof</u> (combinatorial)

when

$$(*) \qquad (x+y)^n = \overset{1}{(x+y)}\,\overset{2}{(x+y)} \cdots\cdots \overset{n}{(x+y)}$$

is expanded, and before like terms are combined, we have 2^n terms of the form.

$$x^{n-k} y^{k} \quad (\text{some } 0 \le k \le n).$$

Fix a particular k. To determine the coef. of $x^{n-k} y^{k}$ (after combining) we count the # of times this term is obtained in the expansion of $*$.

To obtain $x^{n-k} y^{k}$, we choose k factors to contribute a y (and $\therefore$ $n-k$ factors to contribute x).

i.e. we choose k elements from the set $\{1, 2, \ldots, n\}$ of labels.

$$\#\text{ways} = \binom{n}{k}$$

Hence $x^{n-k}y^k$ occurs $\binom{n}{k}$ times in the expansion. $\therefore$ when like terms are combined, the coef. of $x^{n-k}y^k$ is $\binom{n}{k}$. ■

<u>corollary</u>

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

<u>Proof</u>

let $x = y = 1$ in Bin. Thm. ■

Corollary

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof

let $x=1, y=-1$ in Bin. Thm. ▨

Algebraic Proof use Pascal's id:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \qquad (1 \le k \le n)$$

Proof (algebraic)

We show

$$\forall n \geq 0 : \boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k} \quad P(n)$$

I. $P(0)$ says $(x+y)^0 = \binom{0}{0} x^0 y^0$,

i.e. $1 = 1$. ✓

II a. $\forall n \geq 0 : P(n) \to P(n+1)$

Let $n \geq 0$. Assume

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

must show

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k$$

Thus

$$(x+y)^{n+1} = (x+y)(x+y)^n$$

$$= (x+y) \cdot \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= x \cdot \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k} + y \cdot \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^{k} + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n-(k-1)} y^{(k-1)+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n+1-k} y^{k}$$

$$= \binom{n}{0} x^{n+1} y^{0} + \sum_{k=1}^{n} \binom{n}{k} x^{(n+1)-k} y^{k}$$

$$+ \sum_{k=1}^{n} \binom{n}{k-1} x^{(n+1)-k} y^{k} + \binom{n}{n} x^{0} y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^{0} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] x^{(n+1)-k} y^{k}$$

$$+ \binom{n+1}{n+1} x^{0} y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n+1}{k} x^{(n+1)-k} y^k + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k . \qquad \blacksquare$$

<u>Read</u>

- Vandermonde's identity
- corollaries.

<u>Ex.</u>

find coefficients of x^8 and x^{11} in full expansion of

$$(x^2+3)^{20}$$

We have

$$(x^3+3)^2 = \sum_{K=0}^{20} \binom{20}{K} (x^2)^{20-K} \cdot 3^K$$

$$= \sum_{K=0}^{20} 3^K \cdot \binom{20}{K} \cdot x^{40-2K}$$

- $40-2K=8 \Rightarrow 32=2K \rightarrow \boxed{K=16}$

$(\text{coef. of } x^8) = 3^{16} \cdot \binom{20}{16}$

- $40-2K=11 \Rightarrow 29=2K$ X odd $\neq$ even

$(\text{coef. of } x^{11}) = 0$

Exercise

show

$$\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c} = \binom{n}{b}\binom{n-b}{a}\binom{n-a-b}{c}$$

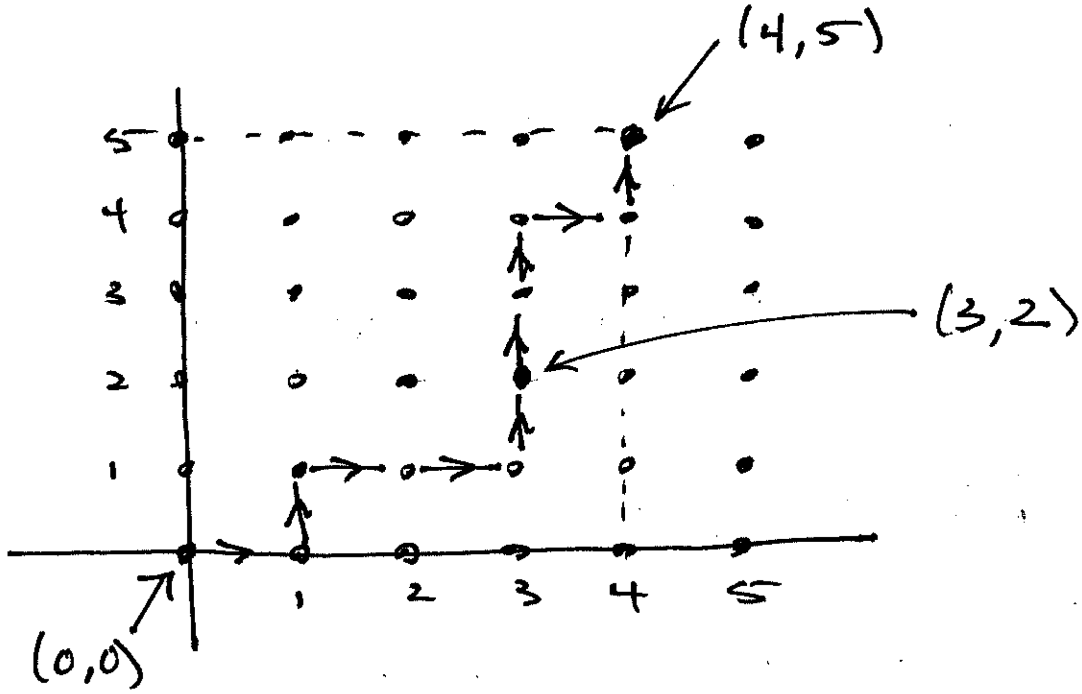
$$= \left[\text{any Permutation of } \{a,b,c\}\right]$$

for $0 \le a+b+c \le n$, $a,b,c,n \in \mathbb{N}$.

<u>Hint</u>: count # strs. of len. n from $\{0,1,2,3\}$ with exactly a 0's, b 1's, c 2's.

Ex. Lattice Paths

$\mathbb{Z} \times \mathbb{Z}$



count the # of Paths in $\mathbb{Z} \times \mathbb{Z}$ from $(0,0)$ to $(4,5)$ consisting of single steps $\rightarrow$ or $\uparrow$ (1 unit).

Each such Path can be encoded as a bit string (on $\{U,R\}$) of length 9 containing exactly 4 R's (or 5 U's)

$$\text{ans} = \binom{9}{4} = \binom{9}{5} = \boxed{126}$$

How many such paths pass through (3,2)?

$$\text{ans} = \binom{5}{3} \cdot \binom{4}{1} = 10 \cdot 4 = \boxed{40}$$

6.5 Generalized Perm. & comb.

<u>Recall</u>

- k-Permutations of an n-set (<u>without</u> repetition).

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- k-Permutations of an n-set <u>with</u> repetition. (i.e. strings)

$$= n^k$$

- k-combinations of an n-set (<u>without</u> repetition). (i.e. subsets)

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$