CSE 16 8-3-23

<u>Notation</u>

$P(n,k)$ = # of k-permutations of an n-set. $(0 \le k \le n)$

<u>Theorem</u>

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

for $0 \le k \le n$

<u>Rmk</u>: if $|S| = n$, an n-permutation of S is just a permutation of S.

$$P(n,n) = n!$$

Defn

A **k-combination** of S is an unordered collection of k of its elements, i.e. a k-element subset of S.

Ex. $S = \{1, 2, 3\}$, $k = 0, 1, 2, 3$

	k-combinations
$k=0$	ϕ
$k=1$	$\{1\}, \{2\}, \{3\}$
$k=2$	$\{1,2\}, \{1,3\}, \{2,3\}$
$k=3$	$\{1,2,3\}$

notation

- $C(n,k) = \#$ of k-combinations of an n-set

also

- $\binom{n}{k} = C(n,k)$ read: 'n-choose-k'

Theorem

for $0 \le k \le n$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Proof

Let S be a set with $|S| = n$.

The task of constructing a k-permutation of S decomposes into 2 subtasks:

- choose a k-subset of S

  $\#$ways $= \binom{n}{k} = C(n,k)$

- choos a permutation of this subset

  $\#$ways $= k!$

By the Product rule

$$P(n,k) = k! \cdot C(n,k)$$

$$\therefore \quad \frac{n!}{(n-k)!} = k! \cdot C(n,k)$$

$$\therefore \quad C(n,k) = \frac{n!}{k!\,(n-k)!}$$

$$\therefore \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \blacksquare$$

<u>Ex.</u>

How many bit strings of length 10 contain exactly 6 1's.

	1		1		1	1	1		1
1	2	3	4	5	6	7	8	9	10

we choose which 6 Positions $\{1, 2, 3, \ldots, 10\}$ are to be occupied by 1's.

$$\therefore \text{ ans} = \binom{10}{6} = \frac{10!}{6!\,4!} = \frac{10\cdot \cancel{9}^{3}\cdot\cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = \boxed{210}$$

could also choose which 4 of positions $\{1, 2, \ldots, 10\}$ will contain 0.

$$\therefore \text{ ans} = \binom{10}{4} = \frac{10!}{4!\,6!} = \boxed{210}.$$

Theorem

for $0 \leq k \leq n$

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof (algebraic)

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}.$$

$\blacksquare$

Two kinds of proofs:

- *algebraic*: manipulate known identities
- *combinatorial*: argue that LHS and RHS count the same set.

Proof (combinatorial)

Let $|S| = n$. Define for $0 \le k \le n$

$$S^{(k)} = \{ k\text{-subsets of } S \} \subseteq \mathbb{P}(S)$$

Define $f : S^{(k)} \to S^{(n-k)}$ by

$$f(A) = \overline{A} = S - A$$

for any $A \in S^{(k)}$. Observe

$$f \circ f(A) = f(f(A)) = f(\bar{A})$$
$$= \bar{\bar{A}} = A$$

$\therefore \quad f \circ f = id \quad \therefore$

$\therefore \quad f$ is invertible with $f^{-1} = f$

$\therefore \quad f$ is a bijection

$\therefore \quad |S^{(k)}| = |S^{(n-k)}|$

$\therefore \quad \binom{n}{k} = \binom{n}{n-k}$. $\blacksquare$

Defn
A ternary string is a string on alphabet $\{0,1,2\}$

Ex.
let $n, k_1, k_2 \in \mathbb{N}$ satisfy $0 \le k_1 + k_2 \le n$

find the # of ternary strings of length n with exactly k_1 0's and k_2 1's.

$$\text{ans} = \binom{n}{k_1} \cdot \binom{n-k_1}{k_2}$$ ✓

choose Positions for 0's — choose Pos. for 1's

or

$$\text{ans} = \binom{n}{k_2} \cdot \binom{n-k_2}{k_1}$$ ✓

choose Pos. for 1's — choose Pos. for 0's note: we've proved

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \binom{n}{k_2}\binom{n-k_2}{k_1}$$

for any $k_1, k_2, n \in \mathbb{N}$ s.t. $0 \le k_1 + k_2 \le n$

Exercise

Give an algebraic proof of this identity.

6.4 Binomial Coefficients

Observe

$$\binom{0}{0}=1$$

$$\binom{1}{0}=1, \quad \binom{1}{1}=1$$

$$\binom{2}{0}=1, \quad \binom{2}{1}=2, \quad \binom{2}{2}=1$$

$$\binom{3}{0}=1, \quad \binom{3}{1}=3, \quad \binom{3}{2}=3, \quad \binom{3}{3}=1$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\begin{array}{c|ccccccccccccc}
\underline{n} & & & & & & & & & & & & & \\
0 & & & & & & & 1 & & & & & & \\
1 & & & & & & 1 & & 1 & & & & & \\
2 & & & & & 1 & & 2 & & 1 & & & & \\
3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\vdots & & & & & & & \vdots & & & & & &
\end{array}$$

$\boxed{\text{Pascal's Triangle}}$

Theorem (Pascal's identity)

(1) $$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \qquad (1 \le k \le n)$$

or

(2) $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad (1 \le k < n)$$

Proof of (1) (algebraic)

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-(k-1))!}$$

$$= \frac{n!\,(n-k+1)}{k!\,(n-k+1)!} + \frac{n!\cdot k}{k!\,(n-k+1)!}$$

$$= \frac{n!\,(n-\cancel{k}+1+\cancel{k})}{k!\,((n+1)-k)!}$$

$$= \frac{(n+1)!}{k!\,((n+1)-k)!} = \binom{n+1}{k} \qquad \blacksquare$$

Proof (combinatorial)

Let T be a set with $|T| = n+1$.

Pick any $x \in T$. Let $S = T - \{x\}$ so that $|S| = n$.

Task: construct a k-subset of T.
We do this task by performing exactly one of the subtask:

- include x : # ways $= \binom{n}{k-1}$
  choose a $(k-1)$-subset of S, then add x.
- exclude x : # ways $= \binom{n}{k}$
  choose a k-subset of S.

By the sum rule

$$(\#\ k\text{-subsets of } T) = \binom{n}{k} + \binom{n}{k-1}$$

$$\therefore \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \quad \blacksquare$$

<u>Theorem</u>

for all $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

<u>Proof</u> (combinatorial)

Both sides count the # of subsets of an n-element set. $\blacksquare$

The Binomial Theorem

Observe

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1\cdot x + 1\cdot y$$
$$(x+y)^2 = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2$$
$$(x+y)^3 = 1\cdot x^3 + 3\cdot x^2 y + 3\cdot xy^2 + 1\cdot y^3$$
$$(x+y)^4 = 1\cdot x^4 + 4\cdot x^3 y + 6\cdot x^2 y^2 + 4\cdot xy^3 + 1\cdot y^4$$
$$\vdots \qquad\qquad \vdots$$

Each term in $(x+y)^n$ is of the form $x^{n-k}\cdot y^k$ $(0 \le k \le n)$, with coefficients from Pascal's triangle.

Theorem (Binomial theorem)

Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Ex. $n=3$

$$(x+y)^3 = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}$$

choose which factors to contribute y.

set	bit str			monomial
ϕ	0	0	0	$xxx = x^3$
$\{3\}$	0	0	1	$xxy = x^2y$
$\{2\}$	0	1	0	$xyx = x^2y$
$\{2,3\}$	0	1	1	$xyy = xy^2$
$\{1\}$	1	0	0	$yxx = x^2y$
$\{1,3\}$	1	0	1	$yxy = xy^2$
$\{1,2\}$	1	1	0	$yyx = xy^2$
$\{1,2,3\}$	1	1	1	$yyy = y^3$