cse 16 8-2-23

<u>Defn</u>

The <u>Euler Totient function</u>

$$\phi : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$$

is defined for $n \in \mathbb{Z}^+$, $\phi(n)$ is the # of #s in $\{1, 2, \ldots, n\}$ that are relatively Prime to n.

<u>Ex.</u> determine $\phi(1000)$.

$$\phi(1000) = 1000 - (\#\text{ of }\#\text{s } \underline{\text{not}} \text{ rel. Prime to } 1000)$$

<u>note</u> a number is not rel. Prime to $1000 = 2^3 \cdot 5^3$ iff it is either div. by 2, or by 5, or both.

(# of #s div. by 2) $= \frac{1000}{2} = 500$

(# of #s div. by 5) $= \frac{1000}{5} = 200$

(# of #s div. by 10) $= \frac{1000}{10} = 100$

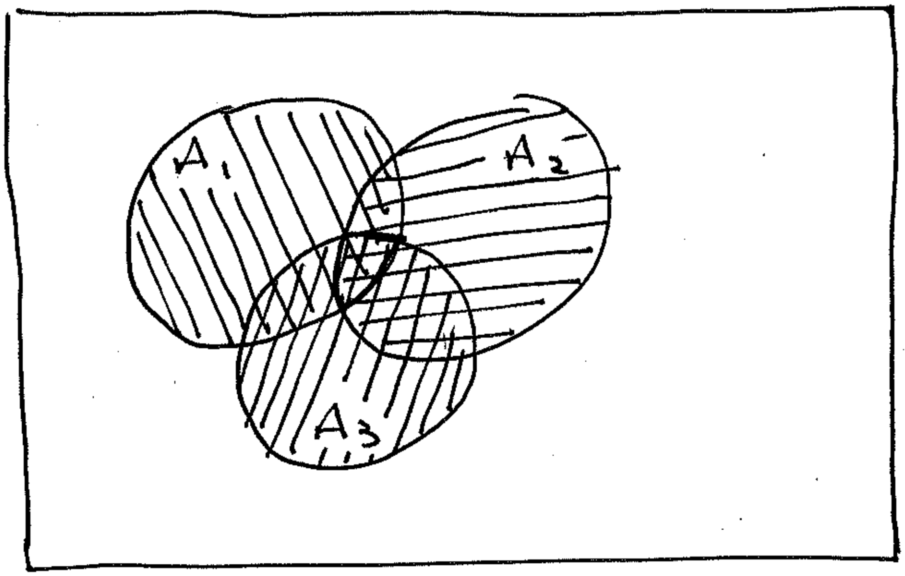
P.I.E

Thus

$$\phi(1000) = 1000 - (500 + 200 - 100) = \boxed{400}$$

Generalizations of PIE

- $$|A_1 \cup A_2 \cup A_3| = (|A_1| + |A_2| + |A_3|) - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$$



- $\displaystyle \left|\bigcup_{i=1}^{4} A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j|$

$$+ \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$

$$- |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$\vdots$

- $$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_i |A_i|$$
$$- \sum_{i<j} |A_i \cap A_j|$$
$$+ \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$
$$\vdots$$
$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

__Ex.__ let p, q be primes, $n \geq 1$, $m \geq 1$. Then

$$\phi(p^n q^m) = p^n q^m - \left(p^{n-1} q^m + p^n q^{m-1} - p^{n-1} q^{m-1}\right)$$
$$= p^n q^m - p^{n-1} q^m - p^n q^{m-1} + p^{n-1} q^{m-1}$$
$$= p^{n-1} q^{m-1} (pq - q - p + 1)$$

$$= p^{n-1} q^{m-1} (p-1)(q-1)$$

$$= (p^n - p^{n-1})(q^m - q^{m-1})$$

$$= \phi(p^n) \cdot \phi(q^m)$$

Exercise

find $\phi(p^n \cdot q^m \cdot r^k)$ where p, q, r are primes, $n \geq 1$, $m \geq 1$ and $k \geq 1$.

6.2 Pigeonhole Principle

Theorem (Pigeonhole Principle, PP)

If $k+1$ (or more) objects are placed in k boxes, then (at least) 1 box contains (at least) 2 objects.

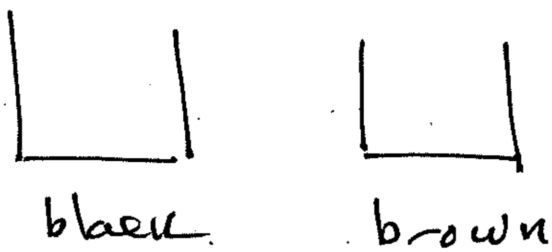
Ex.

A drawer contains 12 black socks & 12 brown socks. A man removes socks in the dark.

- What is the smallest # of socks that must be removed to guarantee a pair of same color.

objects ~ socks

boxes ~ colors



answer: $\boxed{3}$

Aside on $\lfloor \cdot \rfloor$ & $\lceil \cdot \rceil$

floor $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$

ceiling $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$

are defined as follows

- $\lfloor x \rfloor$ = greatest int $\leq x$

  equiv: $n = \lfloor x \rfloor$ iff $n \leq x < n+1$

- $\lceil x \rceil$ = least int $\geq x$

  equiv: $n = \lceil x \rceil$ iff $n-1 < x \leq n$

Alternatively: $\lfloor x \rfloor$ and $\lceil x \rceil$ are the unique integers satisfying

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

<u>Theorem</u> (Generalized P.P., GPP)

If n (or more) objects are placed in k boxes, then (at least) 1 box contains (at least) $\lceil \frac{n}{k} \rceil$ objects.

<u>Proof</u> (Contradiction)

Assume each box contains at most $(\lceil \frac{n}{k} \rceil - 1)$ objects. Then n, the # of objects satisfies

$$n \leq k \cdot \left(\lceil \tfrac{n}{k} \rceil - 1\right) < k \cdot \left(\left(\tfrac{n}{k} + \cancel{1}\right) - \cancel{1}\right) = \cancel{k} \cdot \tfrac{n}{\cancel{k}} = n$$

$\therefore$ $n < n$, a contradiction!!

$\therefore$ some box contains $\lceil \frac{n}{k} \rceil$ (or more) objects. ∎

<u>note</u> we recover P.P. when $n = k+1$, for then

$$\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{k+1}{k} \right\rceil = \left\lceil 1 + \frac{1}{k} \right\rceil = 2$$

<u>Ex</u> sock drawer problem again

- what is smallest # of socks whose removal guarantees 6 of the same color.

say ans. is n

objects ~ socks : n
boxes ~ colors : 2

we seek smallest n s.t.

$$\left\lceil \frac{n}{2} \right\rceil = 6$$

$\therefore \quad 5 < \frac{n}{2} \leq 6 \quad$ (smallest n)

$\therefore \quad \underline{10 < n} \leq 12 \quad$ (smallest n)

$\therefore \quad \boxed{n = 11}$

Ex.

How many students must be enrolled at a Univ. to guarantee at least 100 are born in same month?

Objects ~ Student : n

boxes ~ months : 12

$\left\lceil \frac{n}{12} \right\rceil = 100 \quad$ smallest n

$99 < \frac{n}{12} \leq 100$

$$\underbrace{12 \cdot 99 < n}\ \leq 12 \cdot 100$$

$$\therefore\ n = 12 \cdot 99 + 1 = \boxed{1189}$$

Note: $\lceil \frac{n}{k} \rceil = m$

$n = \#$ objects
$k = \#$ boxes
$m = \#$ guaranteed to have in a box.

Ex.
There are 38 different periods during which classes are scheduled at a Univ. If there are 677 different classes, how many rooms are needed?

objects ~ classes : 677

boxes ~ periods : 38

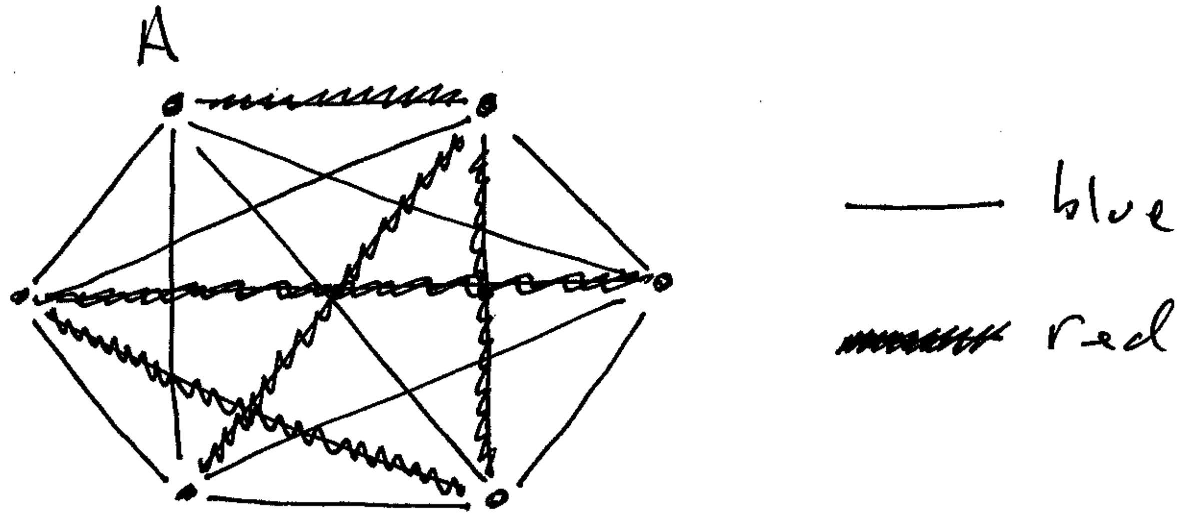
By G.P.P. at least one period has

$$\left\lceil \frac{677}{38} \right\rceil = \lceil 17.8.. \rceil = 18$$

scheduled to it. $\therefore$ at least 18 rooms are necessary.

Theorem

In any group of 6 people in which each pair are either friends or enemies, there must exist either a group of 3 mutual friends or 3 mutual enemies.



Proof

Let A be a person in the group

The remaining 5 fall into 2 classes

$F = \{$friends of $A\}$

$E = \{$enemies of $A\}$

By GPP at least $\lceil \frac{5}{2} \rceil = 3$ people are in the same class.

Case 1 : $|F| \geq 3$

Say B, C, D are friends of A. If any two of these, say $\{B, C\}$ are friends of each other, then $\{A, B, C\}$ are 3 mutual friends. Otherwise $\{B, C, D\}$ are 3 mutual enemies.

Case 2 $|E| \geq 3$

follows from case 1 by swapping

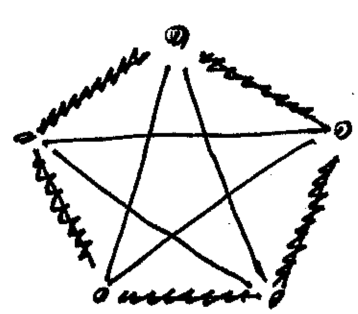
'friend' $\longleftrightarrow$ 'enemy'

In both cases, we have a group of 3 mutual friends or 3 mutual enemies. ∎

Note: 6 is the smallest int for which stmt. is true.

Ex. 5 people

- No Δ of friends
- No Δ of enemies



6.3 Permutations & Combinations

Defn

A permutation of a set is an ordered arrangement of its elements.

Ex. $S = \{1, 2, 3\}$

Permutations of S:

123, 132, 213, 231, 312, 321

Defn

A k-Permutation of a set S is an ordered arrangement of k of its elements $(0 \leq k \leq |S|)$

Ex. $S = \{1, 2, 3\}$, $k = 0, 1, 2, 3$

$k = 0$: ϕ

$k = 1$: 1, 2, 3

$k = 2$: 12, 21, 13, 31, 23, 32

$k = 3$: 123, 132, 213, 231, 312, 321