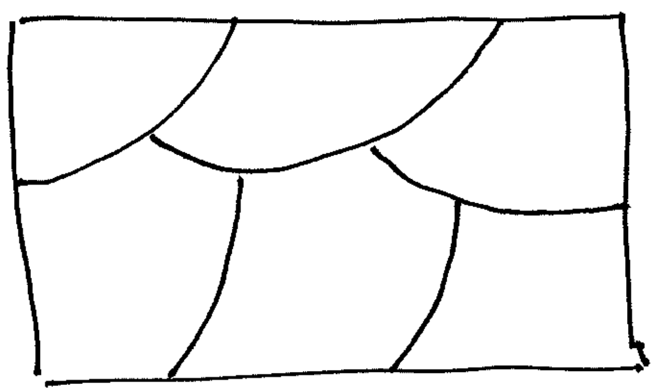
CSE 16 8-17-23

- Final tomorrow: 11:00 am - 12:45 pm

Addition Rule



<u>Partition</u>

- mutually exclusive
- exhaustive

Ex.

find a recurrence for the # of bit strs. of len. n containing '000'.

let C_n = (# such strs.)

$$C_0 = 0$$
$$C_1 = 0$$
$$C_2 = 0$$
$$C_3 = 1 \qquad \{000\}$$
$$\vdots$$

'000'

(1) $\underbrace{x\,x \cdots x}_{n-1}\,1$: $\quad C_{n-1}$

'000'

(2) $\underbrace{x\,x \cdots x}_{n-2}\,10$: $\quad C_{n-2}$

(3) $\underbrace{\overset{\text{'000'}}{xx\cdots x}}_{n-3}100$: C_{n-3}

(4) $\underbrace{\overset{\text{any}}{xx\cdots x}}_{n-3}000$: 2^{n-3}

$$\therefore \; C_n = C_{n-1} + C_{n-2} + C_{n-3} + 2^{n-3}$$

$$C_4 = 1 + 0 + 0 + 2^1 = 3$$

$$C_5 = 3 + 1 + 0 + 2^2 = 8$$

$$C_6 = 8 + 3 + 1 + 2^3 = 20$$

$\vdots$

Exercise:

Find a reccurrence for the #
of ternary strs. (alpha = {0, 1, 2})
of len. n containing '00'

answer:

$$T_0 = 0$$

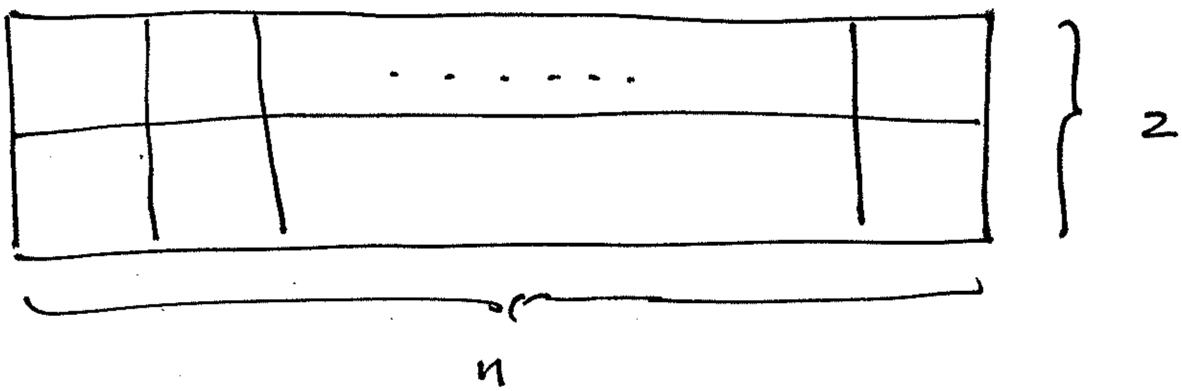
$$T_1 = 0$$

$$T_n = 2T_{n-1} + 2T_{n-2} + 3^{n-2}$$

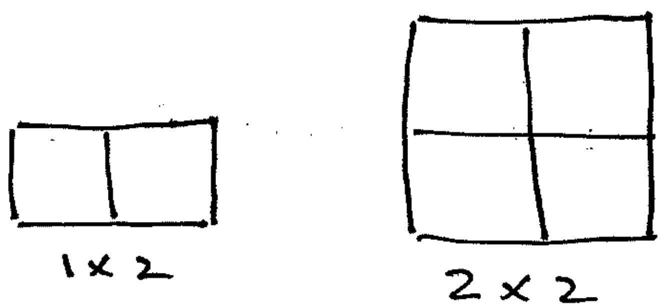
Ex.

In how many ways can a $2 \times n$ walkway be tiled using 1×2 and 2×2 tiles?

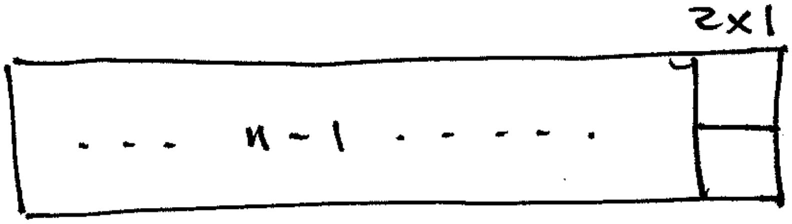
walk



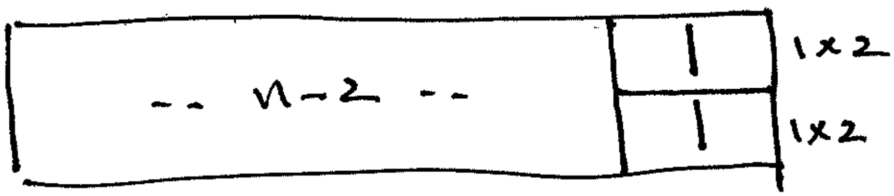
tiles



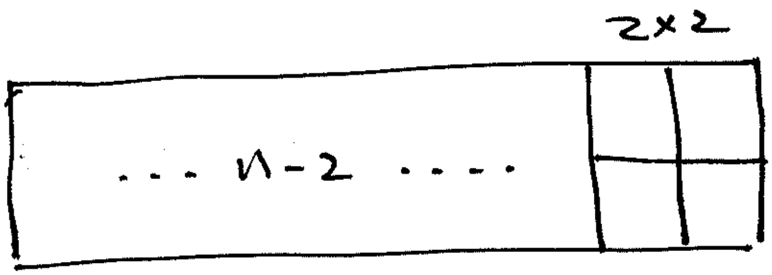
There are 3 mutually exclusive and exhaustive ways the walkway can end.

(1) 

A_{n-1}

(2) 

A_{n-2}

(3) 

A_{n-2}

let A_n denote # of such tilings of len. n. By Sum rule

$$A_n = A_{n-1} + 2A_{n-2} \quad (\text{for } n \geq 2)$$

$A_0 = 1$ (empty)

$A_1 = 1$ ( )

$$\therefore \quad A_2 = 1 + 2 \cdot 1 = 3$$

$$A_3 = 3 + 2 \cdot 1 = 5$$

$$A_4 = 5 + 2 \cdot 3 = 11$$

$$\vdots$$

- what is $A_8 = 171$
- " " $A_{10} = 683$
- find Prob. that a random tiling of len. 10 ends in .

$$\text{Prob} = \frac{171}{683} = \boxed{.2504\ldots}$$

- check $A_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)$

Review

(7.2) # 36

Show if $E_1, \ldots, E_n \subseteq S$ are pairwise disjoint, then

$$P\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} P(E_k).$$

Proof

I. for $n=1$, we have $P(E_1) = P(E_1)$. ✓

II. let $n \geq 1$. Assume

$$P\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} P(E_k).$$

must show

$$P\left(\bigcup_{k=1}^{n+1} E_k\right) = \sum_{k=1}^{n+1} P(E_k)$$

Thus

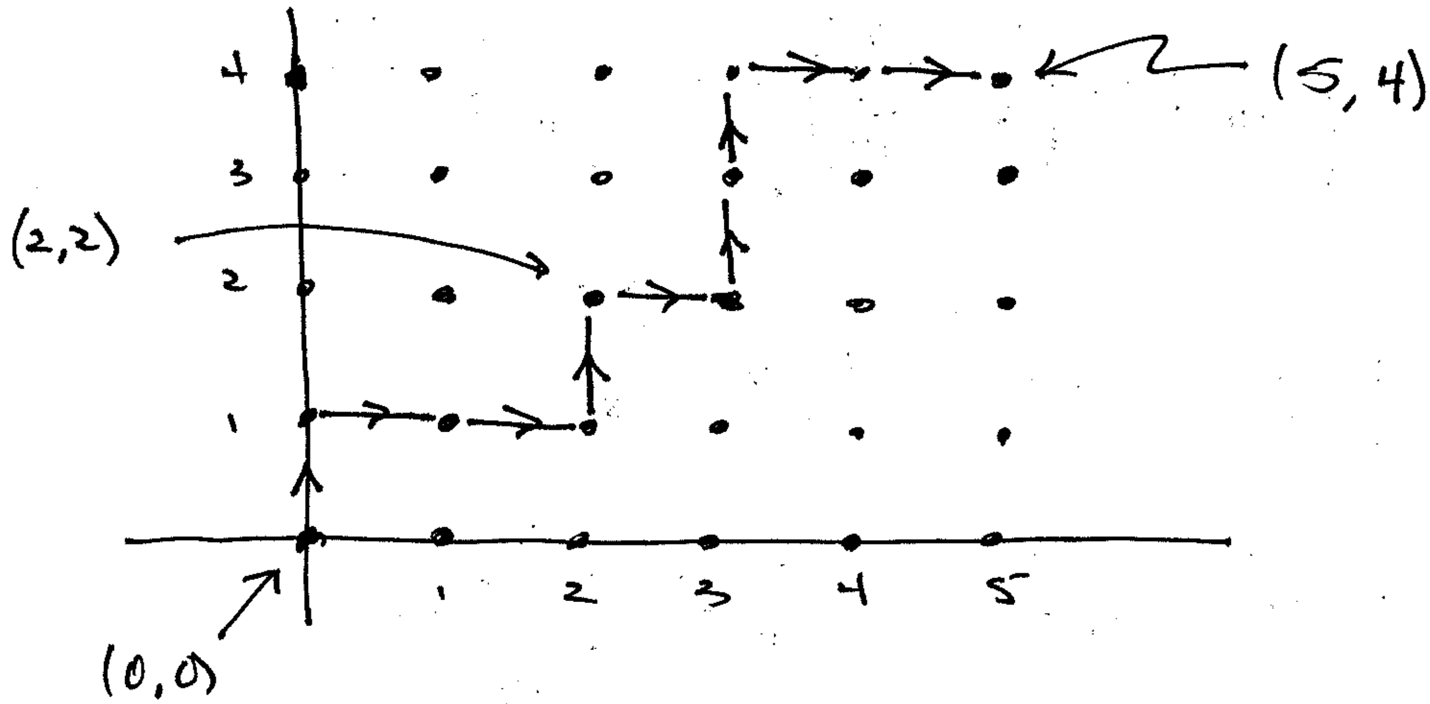
$$P\left(\bigcup_{k=1}^{n+1} E_k\right) = P\left(\left(\bigcup_{k=1}^{n} E_k\right) \cup E_{n+1}\right)$$

$$= P\left(\bigcup_{k=1}^{n} E_k\right) + P(E_{n+1}) \quad \text{by axiom (3)}$$

$$= \sum_{k=1}^{n} P(E_k) + P(E_{n+1}) \quad \text{by ind. hyp.}$$

$$= \sum_{k=1}^{n+1} P(E_k). \qquad \blacksquare$$

Ex. Integer Lattice $\mathbb{Z} \times \mathbb{Z}$



- count # Polygonal Paths from (0,0) to (5,4) using only steps $\{\underset{R}{\rightarrow}, \underset{U}{\uparrow}\}$.

Each such Path can be encoded as a bit string: URRURUURR

i.e. bit strings of len. 9 cont. exactly 4 U's (or 5 R's).

$$\text{answer} = \binom{9}{4} = \frac{9!}{4!\,5!} = \frac{9\cdot\overset{2}{\cancel{8}}\cdot 7\cdot\cancel{6}}{\cancel{4\cdot 3\cdot 2}} = \boxed{126}$$

- how many such Paths Pass through $(2,2)$

$\#$ $(0,0)$ to $(2,2)$: $\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2} = 6$

$\#$ $(2,2)$ to $(5,4)$: $\binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$

$\text{ans} = 6 \cdot 10 = 60$

- what is the Prob. that a random Path Passes through $(2,2)$?

$$\text{Prob.} = \frac{60}{126} = \boxed{.47\ldots}$$

- what is Prob. that Path <u>does not</u> pass through $(1,2)$ and $(4,3)$

(8.1) # 10

let B_n denote # of bit strs. of len. n containing '01'

- let A_n denote # of bit strs. of len. n not cont. '01'

not cont. '01' : no 0's left of any 1's.

∴ all 0's right of all 1's.

$$\left.\begin{array}{l} \overbrace{0\ 0 \cdots\cdots 0}^{n} \\ 1\ 0 \cdots\cdots 0 \\ 1\ 1\ 0 \cdots 0 \\ 1\ 1\ 1\ 0 \cdots 0 \\ \quad\vdots \\ 1\ 1\ 1 \cdots 1\ 0 \\ 1\ 1\ 1 \cdots 1\ 1 \end{array}\right\} A_n = n+1$$

$\therefore \; B_n = 2^n - A_n = 2^n - (n+1)$

$\therefore \; \boxed{B_n = 2^n - n - 1}$

- find rec. rel. for B_n

'01'

(1) $\underbrace{x\,x \cdots\cdots x}_{n-1}\,0 \;:\; B_{n-1}$

(2) $\overbrace{\underbrace{x\,x \cdots\cdots x}_{n-2}}^{\text{any}}\,01 \;:\; 2^{n-2}$

(3) $\overbrace{\underbrace{x \cdots\cdots x}_{n-3}}^{\text{any}}\,01\,1 \;:\; 2^{n-3}$

(4) $\overbrace{\underbrace{x \cdots\cdots x}_{n-4}}^{\text{any}}\,01\,11 \;:\; 2^{n-4}$

$\vdots$

(n) $01\,1 \cdots\cdots 11 \;:\; 2^{n-n} = 2^0 = 1$

By sum rule

$$B_n = B_{n-1} + \sum_{k=0}^{n-2} 2^k$$

$$= B_{n-1} + \frac{2^{n-1} - 1}{2 - 1}$$

$$\therefore \boxed{B_n = B_{n-1} + 2^{n-1} - 1}$$

Also $B_0 = 0$

note: $B_1 = 0$, $B_2 = 1$, ...