CSE 16 8-16-23

- SETs Response : 70.83 %
- HW7 due tonight 10PM (7.2 only)
- Today: finish 7.2, start 8.1,
  cancel ~~9.1~~
- Friday: final exam 11:00 – 12:35 + 10 min
  so 11:00 – 12:45

Ex.

Let $S = \{1, 2, 3, 4, 5\}$. Define

$$P : \mathbb{P}(S) \longrightarrow \mathbb{R}$$

by

x	1	2	3	4	5
$P(x)$	.1	.2	.2	.4	.1

let $E = \{1, 2, 3\}$, $F = \{3, 4, 5\}$, $G = \{1, 5\}$

- are E & F independent?

$P(E) \cdot P(F) = (.5)(.7) = .35$

$P(E \cap F) = .2 \neq .35$

so E, F are <u>not</u> ind.

• are E, G independent?

$$P(E) \cdot P(G) = (.5)(.2) = .1$$

$$P(E \cap G) = .1 = P(E) \cdot P(G)$$

E, G are ind.

<u>Defn</u>

A <u>Bernoulli Trial</u> is an experiment with 2 possible outcomes

$$S = \{0, 1\}$$

1 = success = heads = true

0 = Failure = tails = false

Usually write

$$p = P(\text{Success})$$

$$q = 1 - p = P(\text{Failure})$$

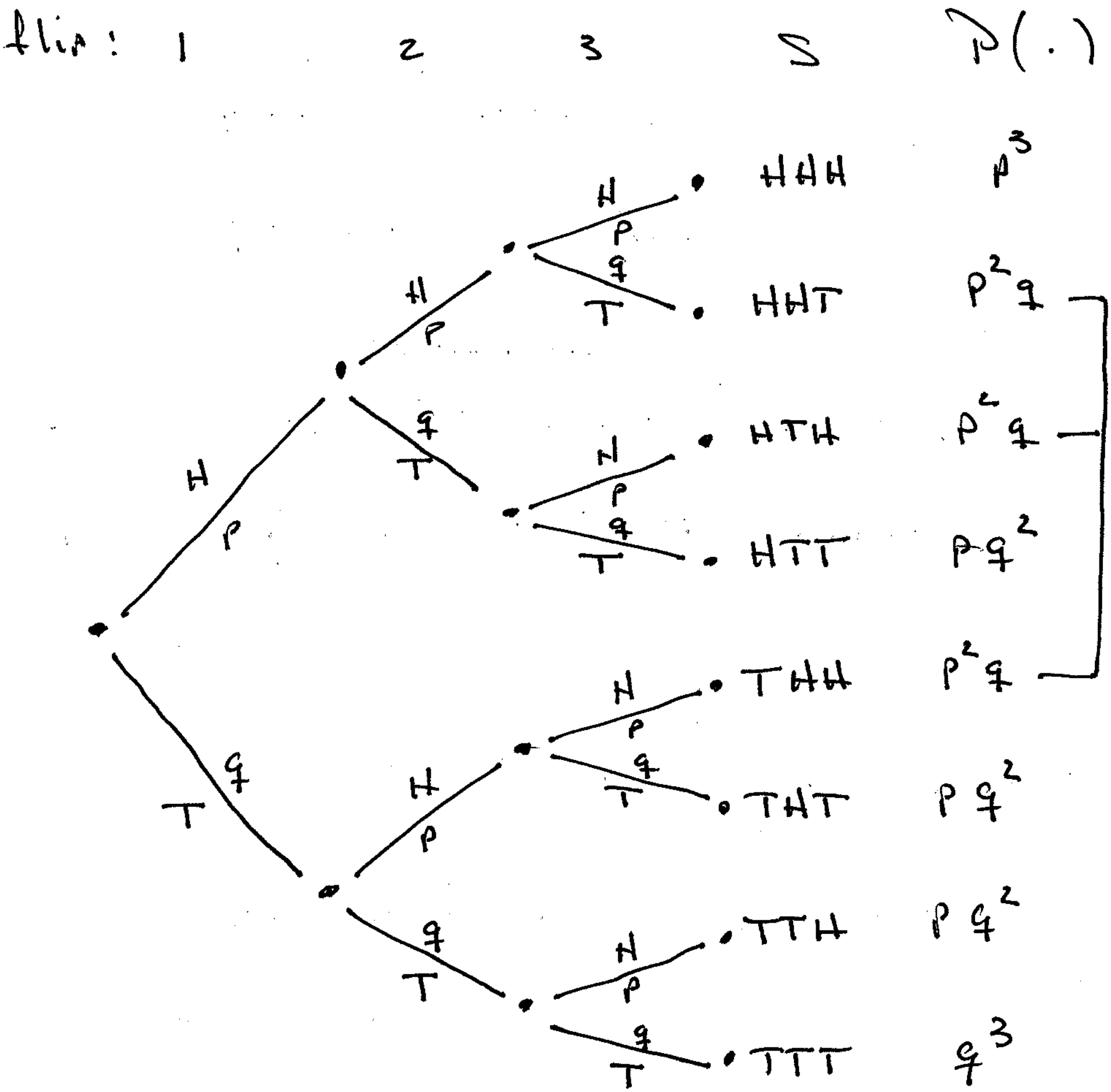
Ex Seq. of n independent Bernoulli trials

A weighted coin is flipped 3 times, where

$$P(\text{Heads}) = p, \quad P(\text{Tails}) = q = 1 - p$$

Assume individual flips are <u>independent</u>.

Tree diagram:



- what is the probability that exactly 2 of 3 flips are heads?

$$P(\#\text{heads}=2) = 3 \cdot p^2 q = 3p^2(1-p)$$

- check that probabilities of all outcomes sum to 1.

$$p^3 + 3p^2 q + 3pq^2 + q^3 = (p+q)^3 = 1^3 = 1 \checkmark$$

<u>Theorem</u>

The probability of exactly k successes in n independent Bernoulli Trials is

$$P(k \text{ succ. in } n \text{ trials}) = \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

given $p = P(\text{success})$ in indiv. trial.

Proof

each outcome is a bit string of length n. We seek Prob. that this str. contains exactly k 1's.

The number of such bit strings is $\binom{n}{k}$. Each such bit string has Probability $p^k q^{n-k}$. $\therefore$ Prob. of k successes in n ind. trials is $\binom{n}{k} \cdot p^k q^{n-k}$. ∎

Ex.

flip a coin 3 times with P(head) = p.

- what's prob. of 2 tails?

$$\binom{3}{1} \cdot p q^2 = \boxed{3 \cdot p(1-p)^2}$$

- at least 2 heads? i.e. (2 heads) or (3 heads)

$$\boxed{\binom{3}{2} \cdot p^2 q + \binom{3}{3} p^3}$$

Ex. flip a coin n times with $P(H) = p$. find Prob. of at least 1 head.

hard way:

$$\text{ans} = \sum_{k=1}^{n} \binom{n}{k} p^k q^{(n-k)}$$

easy way: Prob. of 0 heads is $\binom{n}{0} \cdot p^0 q^{n-0} = 1 \cdot 1 \cdot (1-p)^n = (1-p)^n$

$$\text{ans} = 1 - (1-p)^n$$

Ex.

flip 5 times, $P(H)=p$. find prob. that 1^{st} and last flips are heads.

$$E = \{ 1 \; x \; x \; x \; 1 \}$$

$$\underset{0 \text{ or } 1}{\uparrow \; \uparrow \; \uparrow}$$

note $P(0 \text{ or } 1) = p+q = 1$, so

$$P(E) = P(1) \cdot P(x) P(x) P(x) P(1)$$

$$= p \cdot 1^3 \cdot p = p^2$$

Ex.

A single fair die is tossed we observe whether result is ≤ 4 or ≥ 5

result : $\underbrace{1 \quad 2 \quad 3 \quad 4}_{\text{success}} \quad \underbrace{5 \quad 6}_{\text{failure}}$

$p = \frac{4}{6} = \frac{2}{3}$ $\qquad$ $q = \frac{2}{6} = \frac{1}{3}$

we repeat this 6 times. what is Prob. that half of tosses are ≤ 4.

$$\text{ans.} = \binom{6}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 = \frac{160}{729}$$

$$= \boxed{.2195..}$$

8.1 Applications of recurrence relations

Ex. Let B_n denote the # of bit strings of length n that contain 2 consecutive zeros, i.e. contain substring '00'. Find a recurrence for $(B_n)_{n=0}^{\infty}$.

note:

$B_0 = 0 \quad \phi$

$B_1 = 0 \quad \phi$

$B_2 = 1 \quad \{00\}$

$B_3 = 3 \quad \{000, 001, 100\}$

we construct a bit str of len. n containing '00' by performing <u>exactly one</u> of the following

(1) form a bit str. of len $(n-1)$ containing '00', then append 1.

$$\underbrace{x\, x \cdots\cdots x}_{n-1}\ 1 \quad : \ \#\text{ways} = B_{n-1}$$

(2) form a bit str. of len $(n-2)$ cont. '00', append 10.

$$\underbrace{x\, x \cdots x}_{n-2}\ 10 \quad : \ \#\text{ways} = B_{n-2}$$

(3) form an arbitrary bit str. of len. $n-2$, append 00

$$\underbrace{x\,x \cdots x}_{n-2}\,00 \quad \text{(any)} : \#\text{ways} = 2^{n-2}$$

By the sum rule

$$B_n = B_{n-1} + B_{n-2} + 2^{n-2} \qquad (n \geq 2)$$

so

$$B_0 = 0$$

$$B_1 = 0$$

$$B_2 = 0 + 0 + 2^0 = 1$$

$$B_3 = 1 + 0 + 2^1 = 3$$

$$B_4 = 3 + 1 + 2^2 = 8$$

$B_5 = 8 + 3 + 2^3 = 19$

$B_6 = 19 + 8 + 2^4 = 43$

$\vdots$

- find Prob. that a random bit str. of len. 6 cont. '00'.

$$\text{ans} = \frac{43}{64} = \boxed{.6719\ldots}$$

<u>Exercise</u> show

$$B_n = \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n$$

hint: roots of $x^2 - x - 1 = 0$

$x^2 = x + 1$