CSE 16 8-10-23

A Probility law for a finite sample space S is a function

$$P : \mathbb{P}(S) \to \mathbb{R}$$

In sec. (7.1) we use

$$P(E) = \frac{|E|}{|S|} \quad \Big\} \text{ the uniform Probability law}$$

for any $E \subseteq S$.

Properties

(1) for $E \subseteq S$: $0 \le P(E) \le 1$

(2) $P(\phi) = 0$

(3) $P(S) = 1$

(4) if $E_1, \dots, E_k$ are pairwise disjoint events in S, then

$$P(E_1 \cup \dots \cup E_k) = P(E_1) + \dots + P(E_k)$$

<u>Exercise</u>

Prove (1)-(4) for the uniform law P.

<u>Proof of (4)</u>

$$P(E_1 \cup \dots \cup E_k) = \frac{|E_1 \cup \dots \cup E_k|}{|S|}$$

$$= \frac{|E_1| + \dots + |E_k|}{|S|} \quad \left\{ \begin{array}{l} \text{since } E_i \cap E_j = \phi \\ \text{for } i \neq j \end{array} \right.$$

$$= \frac{|E_1|}{|S|} + \dots + \frac{|E_k|}{|S|} = P(E_1) + \dots + P(E_k)$$

Ex.

a single die is thrown. what is the Probability that the number is at least 5?

$$S = \{1, 2, 3, 4, 5, 6\}, \ |S| = 6$$

$$E = \{5, 6\}, \ |E| = 2$$

$$\therefore P(E) = \frac{2}{6} = \frac{1}{3} = .333\ldots$$

Ex.

Two dice are thrown. what is the Probability that the sum of the numbers is 8?

$$S = \begin{array}{c} \begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 7 \end{array} & \\ \left\{ \begin{array}{cccccc} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{array} \right\} & \begin{array}{c} \leftarrow E \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \end{array}$$

$|S| = 36$, $|E| = 5$

$\therefore P(E) = \frac{5}{36} = .1388\ldots$

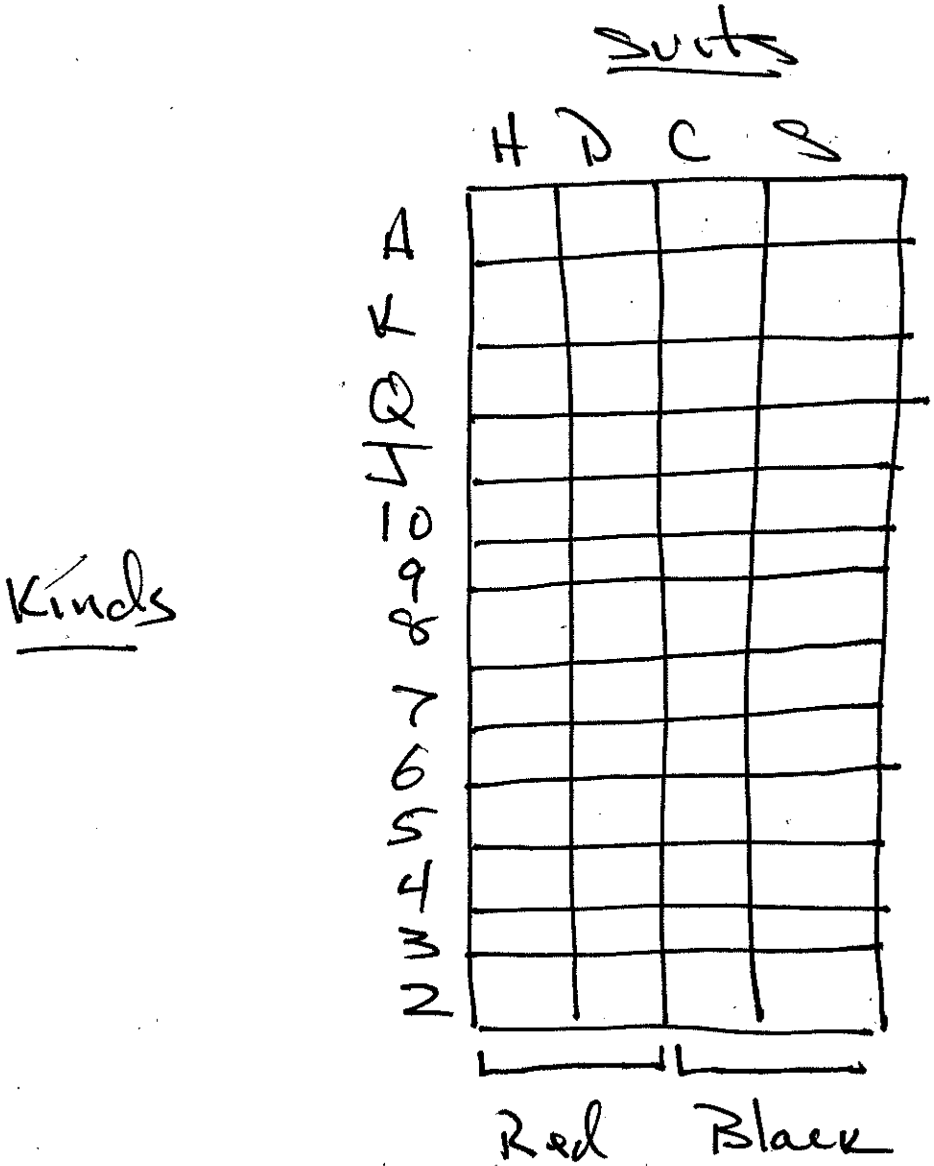
sum	2	3	4	5	6	7	8	9	10	11	12
P(sum)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\uparrow$

highest probability

Ex.

what is the Prob. that a 5-card Poker hand contains 4 of a kind.



$S = \{ \text{5-card Poker hands} \}$

$|S| = \binom{52}{5} = 2{,}598{,}960$

$E = \{\text{hands with 4 of a kind}\}$

$$|E| = \binom{13}{1}\binom{4}{4}\binom{48}{1}$$

$$= 13 \cdot 1 \cdot 48 = 624$$

$$\therefore P(E) = \frac{624}{2{,}598{,}960} = .00024\ldots$$

<u>Ex.</u>
What is the Prob. that a 5-card Poker hand contains 3 of a Kind (and not 4 of a kind).

$E = \{3 \text{ of a Kind}\}$

$$|E| = \binom{13}{1} \cdot \binom{4}{3} \cdot \binom{48}{2} = 58{,}656$$

$$\therefore P(E) = \frac{94}{4165} = .0226\ldots$$

Theorem

Given $E \subseteq S$, the probability of $\overline{E} = S - E$ is

$$P(\overline{E}) = 1 - P(E)$$

Proof.

note $\overline{E} \cap E = \phi$ and $\overline{E} \cup E = S$.

$$\therefore \quad 1 = P(S) = P(E \cup \overline{E}) = P(E) + P(\overline{E})$$

$$\therefore \quad P(\overline{E}) = 1 - P(E) \qquad \blacksquare$$

Ex.

What is the Prob. that a 5-card Poker hand contains at least one ace?

$E = \{\text{hands with} \geq 1 \text{ ace}\}$

$\overline{E} = \{\text{hands with } 0 \text{ aces}\}$

$|\overline{E}| = \binom{48}{5} = 1{,}712{,}304$

$\therefore P(E) = 1 - P(\overline{E})$

$= 1 - \dfrac{1{,}712{,}304}{2{,}598{,}960}$

$= .34115\ldots$

Theorem

Given $E_1, E_2 \subseteq S$,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Proof

divide

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

by $|S|$. $\blacksquare$

Ex.

What is the Prob. that a 5-card Poker hand contains a red ace?

$E_1 = \{$ hand contains AH $\}$ $|E_1| = \binom{51}{4}$

$E_2 = \{$ " " AD $\}$ $|E_2| = \binom{51}{4}$

$E_1 \cap E_2 = \{$ " " both AH and AD $\}$

$|E_1 \cap E_2| = \binom{50}{3}$

$$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{2 \cdot \binom{51}{4} - \binom{50}{3}}{\binom{52}{5}}$$

$$= .18476\ldots$$

Ex.

to win a super lottery you must pick 7 correct numbers out of 40.

$$|S| = \binom{40}{7}, \quad |E| = 1$$

$$P(E) = \frac{1}{18{,}643{,}560} = (5.36..)\,10^{-8}$$

In general

$$P(\text{win}) = \frac{\#\text{ ways of winning}}{\#\text{ ways of playing}}$$

Ex.

In another state the lottery commission picks 12 numbers from $\{1, 2, \ldots, 60\}$. To win you must pick 6 out of these 12.

$$P(\text{win}) = \frac{\binom{12}{6}}{\binom{60}{6}} = (1.85..)10^{-5}$$

Ex.

find the Prob. that a random string of length 10 from $\{a,\dots,z\}$ has no repeated letters.

$S = \{$strs. of len. 10 from $\{a,\dots,z\}\}$

$|S| = 26^{10}$

$E = \{$strs. in S with no repetition$\}$

$$|E| = P(26,10) = \frac{26!}{(26-10)!} = \frac{26!}{16!}$$

$$\therefore P(E) = \frac{26!}{16! \cdot 26^{10}} = .1365\ldots$$

Exercise

find a formula for the Prob. that a random string of length k from an alphabet of size n has no repeated symbols.

<u>note</u>: if $k > n$, then the Prob. is 0 by the Pigeonhole Principle.

7.2 Probability Theory

In general a Probability law

$$P : \mathbb{P}(S) \longrightarrow \mathbb{R}$$

(for a finite sample space S) must satisfy the following axioms.

(1) for all $E \subseteq S$: $P(E) \geq 0$

(2) $P(S) = 1$

(3) (additivity) If $E, F \subseteq S$ are disjoint ($E \cap F = \phi$), then

$$P(E \cup F) = P(E) + P(F).$$

Exercise

Prove

- $P(\phi) = 0$ ✓
- for any $E \subseteq S$: $P(E) \leq 1$
- for $E_1, \ldots, E_k$ pairwise disjoint,

$$P(E_1 \cup \cdots \cup E_k) = P(E_1) + \cdots + P(E_k)$$

Proof that $P(\phi) = 0$:

$$1 = P(S) = P(S \cup \phi) = P(S) + P(\phi)$$

$$\therefore \quad 1 = 1 + P(\phi)$$

$$\therefore \quad P(\phi) = 0$$

Exercise

use axioms (1), (2) and (3) to prove all theorems from (7.1)

Notation

if $x \in S$, then write

$$P(x) = P(\{x\})$$