cse 16 7-28-23

Ex. $\forall n \geq 1$: $\displaystyle\sum_{k=1}^{n} F_{k-1} \cdot F_k = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1}F_{n+1} & (n \text{ odd}) \end{cases}$

Proof

I. $P(1)$ says $F_0 \cdot F_1 = F_0 \cdot F_2$, i.e. $0=0$ ✓

IIb. $\forall n > 1 : P(n-1) \to P(n)$

Let $n > 1$. Assume

$$\sum_{k=1}^{n-1} F_{k-1} \cdot F_k = \begin{cases} F_{n-1}^2 & (n \text{ odd}) \\ F_{n-2}F_n & (n \text{ even}) \end{cases}$$

we must show

$$\sum_{k=1}^{n} F_{k-1} F_k = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases}$$

so

$$\sum_{k=1}^{n} F_{k-1} F_k = \left(\sum_{k=1}^{n-1} F_{k-1} F_k \right) + F_{n-1} \cdot F_n$$

$$= \begin{cases} F_{n-1}^2 + F_{n-1} F_n & (n \text{ odd}) \\ F_{n-2} F_n + F_{n-1} F_n & (n \text{ even}) \end{cases}$$

$$= \begin{cases} F_{n-1}(F_{n-1} + F_n) & (n \text{ odd}) \\ F_n(F_{n-1} + F_{n-2}) & (n \text{ even}) \end{cases}$$

$$= \begin{cases} F_{n-1}F_{n+1} & (n \text{ odd}) \\ F_n \cdot F_n & (n \text{ even}) \end{cases}$$

$$= \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1}F_{n+1} & (n \text{ odd}) \end{cases}$$ ■

<u>Read</u>

- Recursively defined functions
- " " sets & structures
- structural induction

6.1 Basic Counting

Defn

A *string* is a finite sequence of symbols from some alphabet S. Usually $|S| < \infty$.

Ex how many strings of len. 2 from $S = \{a, b, c, \ldots, z\}$ are there?

$$26 \left\{ \underbrace{\begin{array}{llll} aa & ab & \cdots & az \\ ba & bb & \cdots & bz \\ \vdots & \vdots & & \vdots \\ za & zb & \cdots & zz \end{array}}_{26} \right.$$

$$\#\text{strings} = 26 \cdot 26 = 26^2 = \boxed{676}$$

Ex.

how many strings of len. 5 from S.

$$\#\text{choices} = \underset{1}{\underline{26}} \cdot \underset{2}{\underline{26}} \cdot \underset{3}{\underline{26}} \cdot \underset{4}{\underline{26}} \cdot \underset{5}{\underline{26}} = 26^5 = \boxed{11,881,376}$$

Ex.

how many such strings have no repeated symbols?

$$\#\text{strings} = \underset{1}{\underline{26}} \cdot \underset{2}{\underline{25}} \cdot \underset{3}{\underline{24}} \cdot \underset{4}{\underline{23}} \cdot \underset{5}{\underline{22}} = \boxed{7,893,600}$$

Ex.

how many bit strings of length k are there? $(k \geq 0)$

alphabet $= B = \{0, 1\}$

$$\underset{1}{\underline{2}}\ \underset{2}{\underline{2}}\ \underset{3}{\underline{2}}\ \cdots\ \underset{k}{\underline{2}}$$

$$\#\text{ strings} = \underbrace{2 \cdot 2 \cdots 2}_{k\text{ factors}} = 2^k$$

Ex.

how many strs. of len. k from an alphabet of size n are there?

# choices	n	n	n		n
Position:	1	2	3	...	k

$$\# \text{strs} = \boxed{n^k}$$

Ex

how many such strs. have no repeated symbol?

# choices:	n	$(n-1)$	$(n-2)$	....	$(n-k+1)$
Position:	1	2	3		k

$$\# \text{strs} = \boxed{n \cdot (n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}}$$

Rmk

in general

$$(\#\text{strs. w.o. rep.}) = \begin{cases} \dfrac{n!}{(n-k)!} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$

Product Rule

Suppose a task T is decomposed into subtasks $T_1, T_2, \ldots, T_k$ so that after

$$T_1, T_2, \ldots, T_{i-1}$$

are performed, T_i can be performed in n_i ways. Then T can be performed in $(n_1 \cdot n_2 \cdots n_k)$ ways.

Theorem

If A, B are finite sets, so is $A \times B$ and $|A \times B| = |A| \cdot |B|$.

Proof

Task T is to form a pair $(x,y) \in A \times B$

subtasks

T_1 : choose $x \in A$: # ways $= |A|$

T_2 : " $y \in B$: # ways $= |B|$

$\therefore T$ can be performed in $|A| \cdot |B|$,

so $|A \times B| = |A| \cdot |B|$. ∎

Theorem

If S is a finite set, so is $\mathbb{P}(S)$ and $|\mathbb{P}(S)| = 2^{|S|}$.

Proof

the task T is to construct a subset $A \subseteq S$. let

$$S = \{x_1, x_2, \ldots, x_n\}$$

Subtasks:

	choose	# ways
T_1 :	$x_1 \in A$ or $x_1 \notin A$:	2
T_2 :	$x_2 \in A$ or $x_2 \notin A$:	2
$\vdots$		
T_n :	$x_n \in A$ or $x_n \notin A$:	2

$\therefore \quad |\mathbb{P}(S)| = 2^n = 2^{|S|}$.

∎

There is a mapping from

$$\{\text{subsets}\} \xrightarrow{f} \{\text{bit strings}\}$$

<u>Ex</u> let $S = \{1, 2, 3\}$

$$\begin{array}{ccc}
 & f & \\
\emptyset & \longrightarrow & 000 \\
\{1\} & \longrightarrow & 100 \\
\{2\} & \longrightarrow & 010 \\
\{3\} & \longrightarrow & 001 \\
\{1,2\} & \longrightarrow & 110 \\
\{2,3\} & \longrightarrow & 011 \\
\{1,3\} & \longrightarrow & 101 \\
\{1,2,3\} & \longrightarrow & 111
\end{array}$$

$$A \xmapsto{f} b = b_1 b_2 b_3$$

$$b_i = \begin{cases} 0 & i \notin A \\ 1 & i \in A \end{cases}$$

Defn

let A, B be sets. the set of functions with domain and codomain B is denoted

$$B^A = \{f : A \to B\}$$

Theorem

If A, B are finite sets, then so is B^A, and

$$|B^A| = |B|^{|A|}$$

Proof

Task T is to construct a fcn.

$f : A \to B$. suppose $|A| = k$, $|B| = n$

and

$$A = \{x_1, x_2, \ldots, x_k\}$$

T decomposes into subtasks

	choose		# ways
T_1 :	$f(x_1) \in B$	:	n
T_2 :	$f(x_2) \in B$	:	n
$\vdots$			
T_k :	$f(x_k) \in B$	:	n

T can be performed in n^k ways,

so $$|B^A| = n^k = |B|^{|A|}$$

$\blacksquare$

Exercise

find a bijection from B^A to

$\{$ strs. of len. k on alphabet of size $n \}$

Exercise

show that the # of injective fcns. $f: A \to B$, $|A| = k$, $|B| = n$

is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Exercise

see if your bijection in 1^{st} exercise takes injective fcns. to strs. w.o. repetition.

Sum Rule

Suppose task T can be performed by doing exactly one of the subtasks

$$T_1, T_2, \ldots, T_k$$

where each T_i can be performed in n_i ways $(1 \le i \le k)$. Then T can be performed in

$$(n_1 + n_2 + \cdots + n_k)$$

ways.

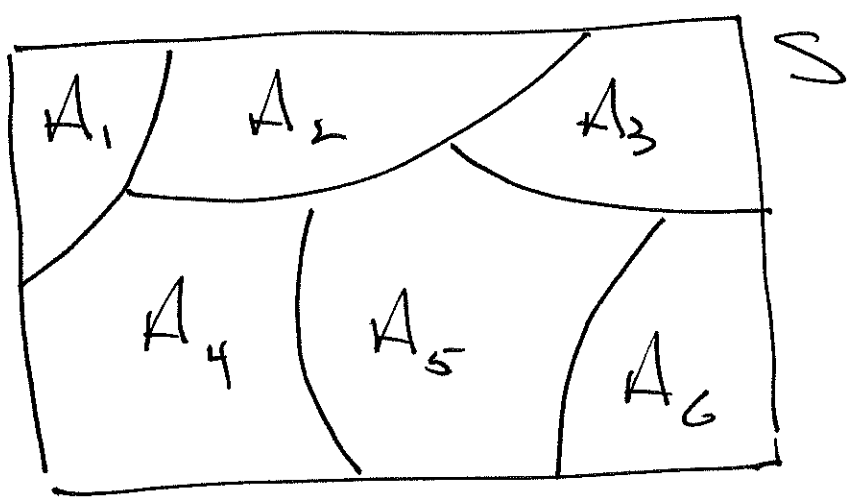
Theorem

Let $A_1, A_2, \ldots, A_k$ be finite sets that are pairwise disjoint (i.e. $A_i \cap A_j = \emptyset$ for any $i \neq j$). Then

$$S = A_1 \cup A_2 \cup \cdots \cup A_k$$

is also finite, and

$$|S| = |A_1| + |A_2| + \cdots + |A_k|$$



Ex.

how many strings of length at most 5 from an alphabet of size 12 are there?

T : form such a string

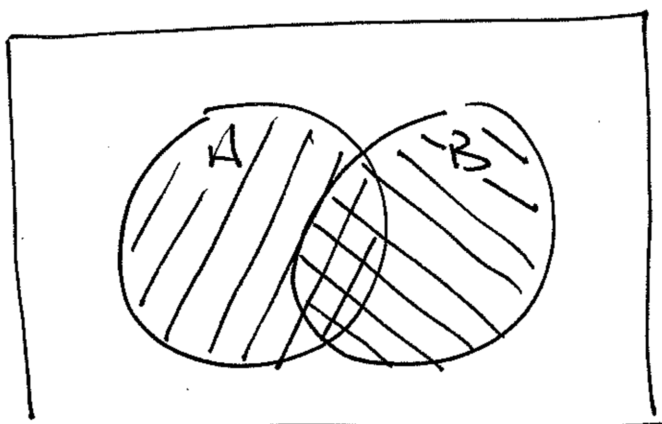
		# ways
T_5	: choose str. of len. 5 :	12^5
T_4	: " " " " 4 :	12^4
T_3	: — — — — — — 3 :	12^3
T_2	: — — — — — — 2 :	12^2
T_1	: — — — — — — 1 :	12^1
T_0	: — — — — — — 0 :	12^0

$$(\#\text{strs.}) = 12^0 + 12^1 + \cdots + 12^5 = \frac{12^6 - 1}{12 - 1}$$

$$= \boxed{271,453}$$

Principle of Inclusion-Exclusion (PIE)

Generalization of sum rule:



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Ex. how many bit strs. of len. 7 either begin '00' or end '111' (or both)?

let $A = \{00xxxxx\}$

$B = \{xxxx111\}$

$A \cap B = \{00xx111\}$

Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 2^5 + 2^4 - 2^2$$

$$= \boxed{44}$$