CSE 16 7-27-23

- no quiz this week
- hw4 due today

Well ordering Property of $\mathbb{Z}^+$

Any non-empty subset of $\mathbb{Z}^+$ contains a least element.

Theorem (PMI)

Given any $P : \mathbb{Z}^+ \to \{F, T\}$, the Proposition

$$(P(1) \wedge \forall n\,(P(n) \to P(n+1))) \to \forall n\, P(n)$$

is true.

Proof

Assume both $P(1)$ and $\forall n\,(P(n) \to P(n+1))$ are true. We must show $\forall n\, P(n)$ is true.

Let $S = \{n \in \mathbb{Z}^+ \mid \neg P(n)\}$, i.e. S is the set of $n \in \mathbb{Z}^+$ for which $P(n)$ is false. It's sufficient to show $S = \emptyset$, for then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Assume (to get a ※) that $S \neq \emptyset$. By the well ordering Prop., S contains a least element, call it m. Thus $\boxed{m \in S}$ and $m-1 \notin S$.

Since $P(1)$ is true, $1 \notin S$, so $m \neq 1$, hence $m-1 \in \mathbb{Z}^+$.

Since $\forall n\,(P(n) \rightarrow P(n+1))$ is true we have, for $n = m-1$, that $P(m-1) \rightarrow P(m)$ is true.

Since $m-1 \notin S$, we have $P(m-1)$ is true.

So we have both

$P(m-1)$ is true

$P(m-1) \rightarrow P(m)$ is true

we conclude $P(m)$ is also true.

Thus $\boxed{m \notin S}$, contradicting $m \in S$.

This contradiction shows our assumption was false, and therefore $S = \phi$. ∎

Variations on induction

We can re-parameterize the induction step.

$$\text{I. } P(1)$$

$$\text{IIb. } \forall n > 1 : \underbrace{P(n-1)}_{\text{ind. hyp}} \rightarrow \underbrace{P(n)}_{\text{ind. conclusion}}$$

original

$$\text{IIa. } \forall n \geq 1 : P(n) \rightarrow P(n+1)$$

(replace n by $n-1$: $n-1 \geq 1 \Leftrightarrow n \geq 2 \Leftrightarrow n > 1$)

Ex. $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$ $\leftarrow P(n)$

Proof

I. same

IIb. $\forall n > 1 : P(n-1) \to P(n)$

let $n > 1$. assume

$$\sum_{k=1}^{n} k = \frac{n(n-1)}{2} .$$

we must show

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} .$$

so

$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k\right) + n$$

$$= \left(\frac{n(n-1)}{2}\right) + n$$

$$= \frac{n(n-1) + 2n}{2}$$

$$= \frac{n(n-1+2)}{2}$$

$$= \frac{n(n+1)}{2}.$$

$\blacksquare$

<u>Exercise</u>

- re-do <u>all</u> previous examples and exercises in form II b.

Another variation is <u>strong induction</u>, or <u>2^{nd} PMI</u>.

It also has 2 Parametrizations

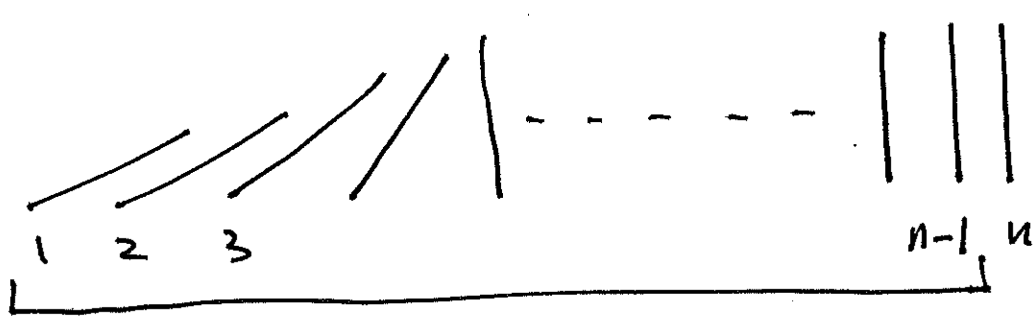
ind. hyp.

I. $P(1)$

IIc. $\forall n \geq 1: \big(P(1) \wedge P(2) \wedge \ldots \wedge P(n)\big) \rightarrow P(n+1)$

I. $P(1)$

IId. $\forall n > 1: \big(P(1) \wedge P(2) \wedge \cdots \wedge P(n-1)\big) \rightarrow P(n)$



we do II d. only

<u>Theorem</u>

Every positive integer can be expressed as a product of (zero or more) primes.

<u>Rmk</u>

This is the existence half of FTA.
The uniqueness half is in 4.3.

<u>Proof</u>

Let $P(n)$ = 'n is a product of primes'

I. $P(1)$ says '1 is a prod. of primes'.
It's the empty product. ✓

IId. $\forall n>1: (P(1) \wedge \dots \wedge P(n-1)) \rightarrow P(n)$.

Let $n>1$. Assume $P(1), \dots, P(n-1)$ are all true, i.e. $P(k)$ is true for all $1 \le k < n$. In other words any k in the range $1 \le k < n$ is a product of primes.

We must show n is a product of primes.

Case 1: n is prime.

In this case $P(n)$ is true and we're done, since n is a prod. of primes.

<u>case 2:</u> n is composite

since $n>1$, we have

$$n = a \cdot b$$

for $1<a<n$ and $1<b<n$. By the induction hypothesis, $P(a)$ and $P(b)$ are true, i.e.

$$a = p_1 p_2 \cdots p_\ell$$

and

$$b = q_1 q_2 \cdots q_m$$

where p_i, q_j $(1 \le i \le \ell, 1 \le j \le m)$ are prime. So

$$n = (p_1 p_2 \cdots p_\ell) \cdot (q_1 q_2 \cdots q_m)$$

showing n is a prod. of primes. ∎

sometimes we need multiple base cases.

P(n)

<u>Ex.</u> $\forall n \geq 2 \; \boxed{\exists x \geq 0, \exists y \geq 0 : n = 2x + 3y}$

<u>Proof.</u>

I. two base cases

$P(2): \; 2 = 2\cdot 1 + 3 \cdot 0$ ✓

$P(3): \; 3 = 2 \cdot 0 + 3 \cdot 1$ ✓

II. d. $\forall n > 3 : (P(2) \wedge P(3) \wedge \dots \wedge P(n-1)) \rightarrow P(n)$

highest base case

lowest base case

Let $n > 3$, i.e. $n \geq 4$. Assume for all k in the range $2 \leq k < n$ that

$$\exists\, x_k \geq 0, \exists\, y_k \geq 0 \;!\; k = 2x_k + 3y_k$$

we must show

$$\exists\, x \geq 0, \exists\, y \geq 0 \;!\; n = 2x + 3y$$

Since $n \geq 4$ we have $2 \leq n-2 < n$. Therefore the induction hypothesis gives $x' \geq 0$ and $y' \geq 0$ s.t.

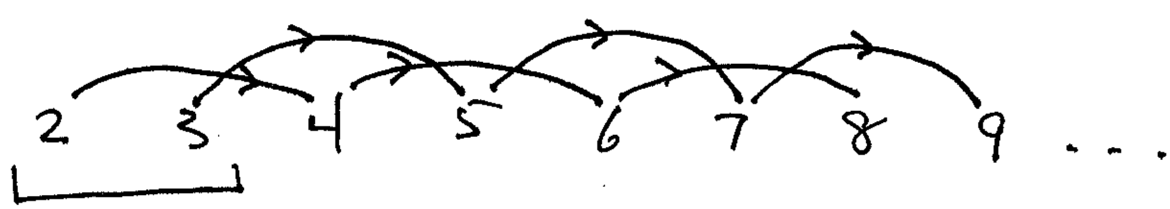
$$n - 2 = 2x' + 3y'$$

Let $x = x' + 1$, $y = y'$. Then $x \geq 0$ and $y \geq 0$ and

$$
\begin{aligned}
n &= (n-2) + 2 \\
&= 2x' + 3y' + 2 \\
&= (2x' + 2) + 3y' \\
&= 2(x'+1) + 3y' \\
&= 2x + 3y .
\end{aligned}
$$

Result follows by 2^{nd} PMI. ▨

Rmk
why did we need 2 base cases?



Base

5.3 Recursive Definitions

Recall Fibonacci Seq.

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \quad (n \geq 2) \end{cases}$$

n	0	1	2	3	4	5	6	..
F_n	0	1	1	2	3	5	8	..

Ex. $\forall n \geq 1$: $\boxed{\sum_{k=1}^{n} F_{2k-1} = F_{2n}}$ $\leftarrow P(n)$

Proof.

I. $P(1)$ is $F_{2\cdot1-1} = F_{2\cdot1}$, i.e.

$$F_1 = F_2 \text{, i.e. } 1 = 1 \quad \checkmark$$

IIa. $\forall n \geq 1$: $P(n) \to P(n+1)$

Let $n \geq 1$. Assume

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n} .$$

We must show

$$\sum_{k=1}^{n+1} F_{2k-1} = F_{2(n+1)}$$

so

$$\sum_{k=1}^{n+1} F_{2k-1} = \left(\sum_{k=1}^{n} F_{2k-1}\right) + F_{2(n+1)-1}$$

$$= F_{2n} + F_{2n+1} \quad \left\{ \begin{array}{l} \text{by ind} \\ \text{hyp.} \end{array} \right.$$

$$= F_{2n+2} \quad \left\{ \begin{array}{l} \text{by the} \\ \text{Fib. recurrence} \end{array} \right.$$

$$= F_{2(n+1)}.$$

Result follows by 1st PMI. ■

<u>Ex.</u>

$$\forall n \geq 1: \sum_{k=1}^{n} F_{k-1} \cdot F_k = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases}$$

Proof

I. $P(1)$ says $F_0 \cdot F_1 = F_0 \cdot F_2$, i.e.

$0 = 0$ ✓

IIb. $\forall n > 1 : P(n-1) \rightarrow P(n)$

Let $n > 1$. assume

$$\sum_{k=1}^{n-1} F_{k-1} \cdot F_k = \begin{cases} F_{n-1}^2 & (n-1 \text{ even}) \\ F_{n-2} F_n & (n-1 \text{ odd}) \end{cases}$$

We must show

$$\sum_{k=1}^{n} F_{k-1} F_k = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases}$$