CSE 16 7-26-23

Goal: Prove a statement

$$* \qquad \forall n \; P(n)$$

for some Prop. fcn. $P: \mathbb{Z}^+ \to \{F, T\}$.
i.e. $P(n) = T$, for all $n \in \mathbb{Z}^+$.

A Proof by mathematical induction has 2 steps.

I. base: Prove $P(1)$ is true.
II. induction: Prove $\forall n \; (P(n) \to P(n+1))$
i.e. let $n \geq 1$ be arbitrary. Assume $P(n)$ is true, show $P(n+1)$ follows.

when I. and II. are complete, we conclude $*$ is true.

Rmks

- $P(n)$ is called the <u>induction hypothesis</u>.
- $P(n+1)$ may be called the <u>induction conclusion</u>.
- induction is not circular reasoning. We assume only $P(n)$ for one particular n.
- like an infinite proof:

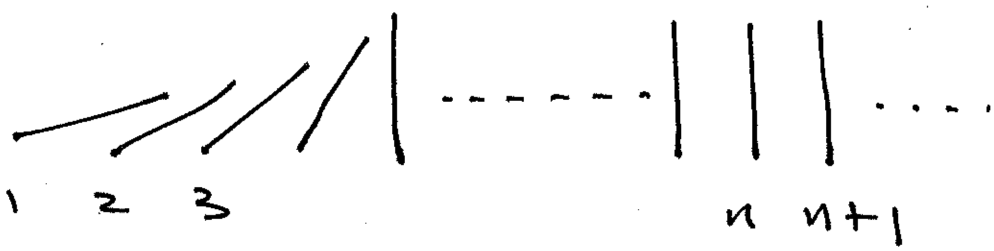
1.	$P(1)$	By I
2.	$P(1) \Rightarrow P(2)$	By II with $n=1$
3.	$P(2)$	by (1) and (2)
4.	$P(2) \Rightarrow P(3)$	by II with $n=2$

5. $P(3)$ by (3) and (4)

6. $P(3) \Rightarrow P(4)$ by II with $n=3$

$\vdots$

- Domino analogy :



- $P(n)$ = 'nth domino falls'

I. $P(1)$ = '1st domino falls'

II. $\forall n \, (P(n) \Rightarrow P(n+1))$

'if any domino falls, then the next one falls'.

<u>conclude</u>: $\forall n \, P(n)$

'all dominos fall'.

- <u>Theorem</u> (Principle of mathematical induction) (PMI)

for any propositional function

$$P : \mathbb{Z}^+ \to \{F, T\}$$

the following is true

$$\left[P(1) \land \forall n \left(P(n) \to P(n+1)\right)\right] \to \forall n\, P(n).$$

Proof later.

<u>Ex.</u> $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} (2k-1) = n^2}$

$P(n)$

Proof

I. P(1) is the statement

$$\sum_{k=1}^{1} (2k-1) = 1^2$$

i.e. $\quad 2\cdot 1 - 1 = 1^2$

i.e. $\quad 1 = 1$

which is true.

II. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$ be chosen arbitrarily.

Assume $P(n)$ is true, i.e. assume

$$\sum_{k=1}^{n} (2k-1) = n^2$$

we must show $P(n+1)$ is true, i.e.

$$\sum_{k=1}^{n+1} (2k-1) = (n+1)^2.$$

so

$$\sum_{k=1}^{n+1} (2k-1) = \sum_{k=1}^{n} (2k-1) + \left(2(n+1)-1\right)$$

$$= n^2 + (2n+2-1) \quad \left\{ \begin{array}{l} \text{by the} \\ \text{induction} \\ \text{hypothesis} \end{array} \right.$$

$$= n^2 + 2n + 1$$

$$= (n+1)^2.$$

$\therefore$ $P(n+1)$ is true.

$\therefore$ $\forall n \geq 1 : \sum_{k=1}^{n} (2k-1) = n^2$ is true

By the P.M.I.

Ex. Let $x \in \mathbb{R}, x \neq 1$. Prove

$$\forall n \geq 1 : \boxed{\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}} \leftarrow P(n)$$

Proof

I. $P(1)$ says: $x^0 = \frac{x^1 - 1}{x - 1}$, i.e. $1 = 1$ ✓

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

Let $n \geq 1$. Assume for this n that

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}.$$

We must show

$$\sum_{k=0}^{(n+1)-1} x^k = \frac{x^{n+1} - 1}{x - 1}.$$

so

$$\sum_{k=0}^{n} x^k = \left(\sum_{k=0}^{n-1} x^k\right) + x^n$$

$$= \frac{x^n - 1}{x-1} + x^n \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \frac{x^n - 1 + x^n(x-1)}{x-1}$$

$$= \frac{\cancel{x^n} - 1 + x^{n+1} - \cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1} - 1}{x-1}.$$

Result follows by P.M.I. ■

Rmks

Always:

- let $n \geq 1$ be arbitrary
- state ind. hypothesis explicitely.
- " " conclusion " .
- state point(s) at which the ind. hyp. is used.

Ex. $\forall n \geq 1$: $\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$. $\leftarrow P(n)$

Proof

I. $P(1)$ is: $1 = \frac{1(1+1)}{2}$, i.e. $1 = 1$ ✓

II. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$. Assume

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

we must show

$$\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2}.$$

so

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \Big\{ \text{by the ind. hyp.}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}.$$

Result follows by PMI. ∎

Ex $$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}} \leftarrow P(n)$$

for all $n \geq 1$.

Proof

I. $P(1)$ says: $1^2 = \frac{1\cdot(1+1)(2\cdot 1+1)}{6}$,

i.e. $1^2 = \frac{1\cdot 2\cdot 3}{6}$, i.e. $1 = 1$.

II. $\forall\, n \geq 1 : P(n) \rightarrow P(n+1)$.

Let $n \geq 1$. Assume

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

must show

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

so

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$\vdots$ exercise: do algebra

$$= \frac{(n+1)(n+2)(2n+3)}{6} .$$

■

Exercise show

$$\forall n \geq 1 : \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

To Prove: $\forall n \geq n_0 : P(n)$

do

I. show $P(n_0)$ is true

II. show $\forall n \geq n_0 : P(n) \Rightarrow P(n+1)$

Ex. $\forall n \geq 4 \quad \boxed{5n+8 < n^2+4n+1} \leftarrow P(n)$

note: $P(1), P(2), P(3)$ are all false

Proof:

I. $P(4)$ says $5 \cdot 4 + 8 < 4^2 + 4 \cdot 4 + 1$,

i.e. $28 < 33$, which is true.

II. $\forall n \geq 4: P(n) \to P(n+1)$.

Let $n \geq 4$. Assume

$$5n+8 < n^2+4n+1.$$

we must show

$$5(n+1)+8 < (n+1)^2+4(n+1)+1.$$

Then

$$\begin{aligned}
5(n+1)+8 &= 5n+5+8 \\
&= (5n+8)+5 \\
&< (n^2+4n+1)+5 && \left\{ \text{by the ind. hyp.} \right. \\
&= n^2+4n+6 \\
&< n^2+4n+6+2n && \left\{ \begin{array}{l} n \geq 4 \to 2n \geq 8 \\ \to 2n > 0 \end{array} \right. \\
&= n^2+6n+6
\end{aligned}$$

by some algebra

$$= (n+1)^2 + 4(n+1) + 1 .$$

Result follows by P.M.I. $\blacksquare$

<u>Rmk</u>

How did we know to add $2n$?

$$\left((n+1)^2 + 4(n+1) + 1\right) - \left(n^2 + 4n + 6\right)$$

$$= \cancel{n^2} + 2n + \cancel{1} + \cancel{4n} + \cancel{4} + \cancel{1} - \cancel{n^2} - \cancel{4n} - \cancel{6} = 2n$$

Ex. let $A_1, A_2, A_3, \ldots$ be subsets of U. Then

$$\forall n \geq 1 : \boxed{\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}}$$

Proof

I. $P(1)$ says $\overline{A_1} = \overline{A_1}$, which is true.

II. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$.

let $n \geq 1$. assume

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$$

must show

$$\overline{\bigcup_{k=1}^{n+1} A_k} = \bigcap_{k=1}^{n+1} \overline{A_k}$$

> 0

$$\overline{\bigcup_{k=1}^{n+1} A_k} = \overline{\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}}$$

$$= \left(\overline{\bigcup_{k=1}^{n} A_k}\right) \cap \overline{A}_{n+1} \quad \left\{ \begin{array}{l} \text{by 2-set} \\ \text{DeMorgan} \end{array} \right.$$

$$= \left(\bigcap_{k=1}^{n} \overline{A}_k\right) \cap \overline{A}_{n+1} \quad \left\{ \begin{array}{l} \text{by ind.} \\ \text{hyp.} \end{array} \right.$$

$$= \bigcap_{k=1}^{n+1} \overline{A}_k .$$

Exercises

- $\forall n \geq 1: \overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}$

Let $P_1, P_2, P_3, \ldots$ be Propositions. Then

- $\forall n \geq 1: \neg\left(\bigwedge_{k=1}^{n} P_k\right) \equiv \bigvee_{k=1}^{n} (\neg P_k)$

- $\forall n \geq 1: \neg\left(\bigvee_{k=1}^{n} P_k\right) \equiv \bigwedge_{k=1}^{n} (\neg P_k)$

5.2 Strong induction & Well ordering.

The Proof of the PMI uses

Well ordering Property of $\mathbb{Z}^+$

Any non-empty subset of $\mathbb{Z}^+$ contains a least element.

Rmks

this is false if we replace

- $\mathbb{Z}^+$ by $\mathbb{Z}$ (or $\mathbb{Z}^-$)
- 'least' by 'greatest'