cse 16 7-14-23

- quizz today

Ex.
Recall: Lucas seq.

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \ L_1 = 1.$$

let $A = \left(\frac{1+\sqrt{5}}{2}\right)$, $B = \left(\frac{1-\sqrt{5}}{2}\right)$,

solutions to $x^2 - x - 1 = 0$, i.e. $x^2 = x + 1$

Thus $A^2 = A + 1$, $B^2 = B + 1$

solution to Lucas: $\boxed{L_n = A^n + B^n}$

check:

$$RHS = L_{n-1} + L_{n-2} = \left(A^{n-1} + B^{n-1}\right) + \left(A^{n-2} + B^{n-2}\right)$$

$$= \left(A^{n-1} + A^{n-2}\right) + \left(B^{n-1} + B\right)^{n-2}$$

$$= A^{n-2}(A+1) + B^{n-2}(B+1)$$

$$= A^{n-2} \cdot A^2 + B^{n-2} \cdot B^2$$

$$= A^n + B^n$$

$$= L_n = \text{LHS}.$$ ∎

2.5 Cardinality of Sets

Defn

Two sets A, B (not necessarily finite) are said to have the same cardinality iff there exists a bijection

$$f : A \to B$$

notation: $|A| = |B|$

Ex. $|\mathbb{N}| = |\mathbb{Z}^+|$

$$\begin{array}{ccccccc} \mathbb{N} = \{ & 0, & 1, & 2, & 3, & \ldots & \} \\ f & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \mathbb{Z}^+ = \{ & 1, & 2, & 3, & 4, & \ldots & \} \end{array}$$

$f(n) = n + 1$

Let $E = \{\text{even, non-negative ints}\}$

$O = \{\text{odd, positive ints}\}$

<u>Ex.</u> $|\mathbb{N}| = |E|$

$f : \mathbb{N} \to E$, $f(n) = 2n$

<u>Ex.</u> $|\mathbb{N}| = |O|$

$f : \mathbb{N} \to O$, $f(n) = 2n+1$

<u>Ex.</u> $|\mathbb{N}| = |\mathbb{Z}|$

$$f(n) = \begin{cases} -\frac{n}{2} & (n \text{ even}) \\ \frac{n+1}{2} & (n \text{ odd}) \end{cases}$$

$$\begin{array}{ccc} \mathbb{N} & f & \mathbb{Z} \\ 0 & \to & 0 \\ 1 & \to & 1 \\ 2 & \to & -1 \\ 3 & \to & 2 \\ 4 & \to & -2 \\ & \vdots & \end{array}$$

Exercise

Prove that if $|A|=|B|$ and $|B|=|C|$ then $|A|=|C|$.

Defn

A set S is called countable iff either

- S is finite

or

- $|S|=|\mathbb{N}|$ (or $|\mathbb{Z}^+|$, or $|\mathbb{Z}|$)

In the second case we call S countably infinite.

Ex $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^+, E, O \ldots$ are countable.

A set that is not countable is called <u>uncountable</u>.

Ex. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable

$$
\begin{array}{ccccc}
11 & 12 & 13 & 14 & \cdots \\
21 & 22 & 23 & 24 & \cdots \\
31 & 32 & 33 & 34 & \cdots \\
41 & 42 & 43 & 44 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{Z}^+ & f & \mathbb{Z}^+ \times \mathbb{Z}^+ \\
1 & \longrightarrow & 11 \\
2 & \longrightarrow & 21 \\
3 & \longrightarrow & 12 \\
4 & \longrightarrow & 31 \\
 & \vdots &
\end{array}
$$

(see (2.5) #31 p.177 for formula).

Exercise

- show that any subset of a countable set is countable. (Hint: suff. to show any subset $\mathbb{Z}^+$ is countable.)
- show that $\mathbb{Q}$ is countable (Hint: find a bijection from $\mathbb{Q}^+$ to a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$.)

Theorem

$\mathbb{R}$ is uncountable.

Proof (contradiction)

Assume $\mathbb{R}$ is countable. Then so is the set

$$S = (0,1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\},$$

and hence S is the range of a seq.

$$S = \{r_1, r_2, r_3, r_4, \ldots\}$$

Each element of S has a unique*
decimal expansion.

$$r_1 = .d_{11}\, d_{12}\, d_{13} \ldots$$
$$r_2 = .d_{21}\, d_{22}\, d_{23} \ldots$$
$$r_3 = .d_{31}\, d_{32}\, d_{33} \ldots$$

where each $d_{ij} \in \{0, 1, 2, \ldots, 9\}$.

* If a number has two dec. expansions, choose the on with a tail of all 9's (not all 0's).
i.e. choose: .4999... instead of .50000....

Define $x \in S$ by

$$x = .c_1 c_2 c_3 \ldots\ldots$$

where
$$c_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

observe $c_i \neq d_{ii}$ for $i = 1, 2, 3, \ldots$,
therefore $x \neq r_i$ for any $i = 1, 2, \ldots$.

Hence $x \notin S$. But clearly $0 < x \leq 1$, so $x \in S$. This x shows our assumption was false. Therefore $\mathbb{R}$ is uncountable. ∎