

CS 2 16 7-14-23

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• quiz 2 today

Ex.

Recall! Lucas seq.

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \quad L_1 = 1$$

$$\text{let } A = \left(\frac{1+\sqrt{5}}{2}\right), \quad B = \left(\frac{1-\sqrt{5}}{2}\right)$$

solutions to $x^2 - x - 1 = 0$, i.e. $x^2 = x + 1$

$$\text{Thus } A^2 = A + 1, \quad B^2 = B + 1$$

$$\text{solution to Lucas: } \boxed{L_n = A^n + B^n}$$

check!

$$\begin{aligned} \text{RHS} &= L_{n-1} + L_{n-2} = (A^{n-1} + B^{n-1}) + (A^{n-2} + B^{n-2}) \\ &= (A^{n-1} + A^{n-2}) + (B^{n-1} + B^{n-2}) \end{aligned}$$

$$= A^{n-2}(A+1) + B^{n-2}(B+1)$$

$$= A^{n-2} \cdot A^2 + B^{n-2} \cdot B^2$$

$$= A^n + B^n$$

$$= L_n = L_{n+2}.$$



2.5 Cardinality of sets

Defn

Two sets A, B (not necessarily finite) are said to have the same cardinality iff there exists a bijection

$$f: A \rightarrow B$$

notation: $|A| = |B|$

Ex. $|\mathbb{N}| = |\mathbb{Z}^+|$

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

$$f \begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{Z}^+ = \{1, 2, 3, 4, \dots\} \end{matrix}$$

$$f(n) = n+1$$

Let $E = \{ \text{even, non-negative ints} \}$

$O = \{ \text{odd, positive ints} \}$

Ex. $|\mathbb{N}| = |E|$

$f: \mathbb{N} \rightarrow E, f(n) = 2n$

Ex. $|\mathbb{N}| = |O|$

$f: \mathbb{N} \rightarrow O, f(n) = 2n + 1$

Ex. $|\mathbb{N}| = |\mathbb{Z}|$

$$f(n) = \begin{cases} -\frac{n}{2} & (n \text{ even}) \\ \frac{n+1}{2} & (n \text{ odd}) \end{cases}$$

\mathbb{N}	f	\mathbb{Z}
0	\mapsto	0
1	\mapsto	1
2	\mapsto	-1
3	\mapsto	2
4	\mapsto	-2
	\vdots	

Exercise

Prove that if $|A| = |B|$ and $|B| = |C|$
then $|A| = |C|$.

Defn

A set S is called countable if
either

- S is finite

or

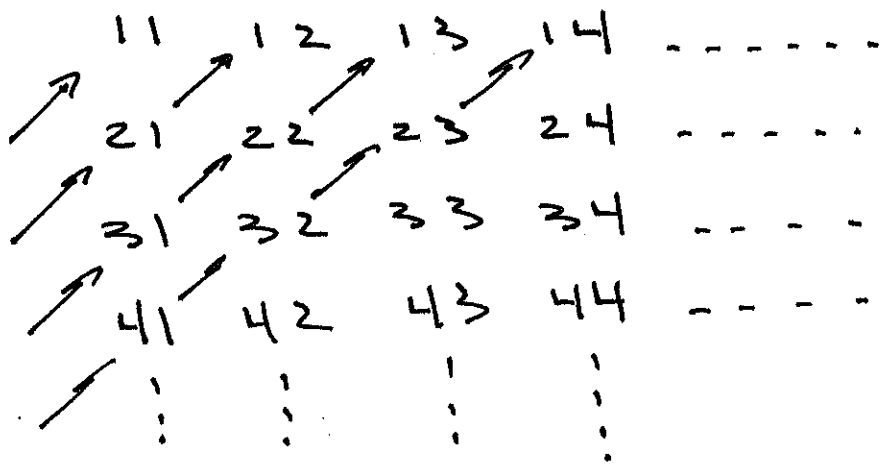
- $|S| = |\mathbb{N}|$ (or $|\mathbb{Z}^+|$, or $|\mathbb{Z}|$)

In the second case we call S countably infinite.

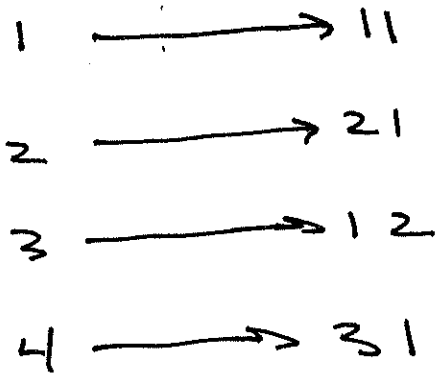
Ex $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{E}, \mathbb{O} \dots$ are countable.

A set that is not countable is called uncountable.

Ex. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable



$\mathbb{Z}^+ \ni \mathbb{Z}^+ \times \mathbb{Z}^+$



⋮

(see (2.5) #31 p.177 for formula).

Exercise

- show that any subset of a countable set is countable. (Hint: soft. to show any subset \mathbb{Z}^+ is countable.)
- show that \mathbb{Q} is countable (Hint: find a bijection from \mathbb{Q}^+ to a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$.)

Theorem

\mathbb{R} is uncountable.

Proof (contradiction)

Assume \mathbb{R} is countable. Then so is the set

$$S = (0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\},$$

and hence S is the range of a seq.

$$S = \{r_1, r_2, r_3, r_4, \dots\}$$

Each element of S has a unique* decimal expansion.

$$r_1 = 0 \cdot \textcircled{d_{11}} d_{12} d_{13} \dots$$

$$r_2 = 0 \cdot d_{21} \textcircled{d_{22}} d_{23} \dots$$

$$r_3 = 0 \cdot d_{31} d_{32} \textcircled{d_{33}} \dots$$

where each $d_{ij} \in \{0, 1, 2, \dots, 9\}$.

* If a number has two dec. expansions, choose the one with a tail of all 9's (not all 0's).
i.e. choose: $.4999\dots$ instead of $.5000\dots$

Define $x \in \mathcal{Q}$ by

$$x = .c_1 c_2 c_3 \dots$$

where

$$c_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

observe $c_i \neq d_{ii}$ for $i = 1, 2, 3, \dots$,
therefore $x \neq r_i$ for any $i = 1, 2, \dots$.

Hence $x \notin \mathcal{Q}$. But clearly $0 < x \leq 1$, so $x \in \mathcal{Q}$. This $\cdot \times$.

shows our assumption was false.
Therefore \mathbb{R} is uncountable.

