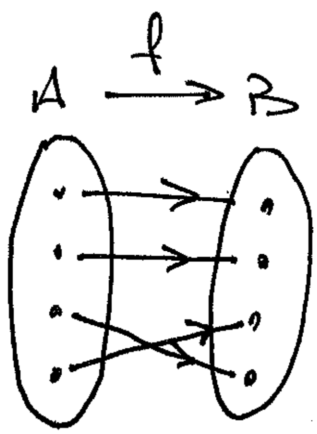
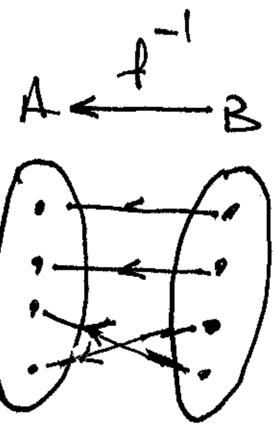
CSE 16 7-13-23

- hw2 due tonight
- quizz tomorrow

Picture





Exercise Prove

(a) if f is invertible, then so is f^{-1} and $(f^{-1})^{-1} = f$.

(b) if is bijective, then so is f^{-1}.

Ex. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$ is invertible and

$$g^{-1}(x) = \frac{x-3}{2}.$$

Ex $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not invertible

Ex. $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$ is invertible.

$$f^{-1}(x) = \sqrt{x}$$

2.4 Sequences and summations

Defn

A *sequence* is a function whose domain is a subset of $\mathbb{Z}$, usually $\mathbb{N}$ or $\mathbb{Z}^+$.

Rmks

- we write a_n instead of $a(n)$ for the image of n under a
- we call a_n the n^{th} term of seq.
- Book writes $\{a_n\}$. we write

$$(a_n) = (a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, \ldots\ldots)$$

- note if a seq. has finite length n, then

$$(a_k)_{k=1}^{n} = (a_1, a_2, \ldots, a_n)$$

is just an orded n-tuple.

<u>EX</u>

- $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

- $(2^n)_{n=0}^{\infty} = (1, 2, 4, 8, 16, \ldots)$

- $(n^2)_{n=3}^{6} = (9, 16, 25, 36)$

- $(-1+4n)_{n=0}^{\infty} = (-1, 3, 7, 11, \ldots)$

Defn

- A <u>geometric sequence</u> is of the form

$$(a\cdot r^n)_{n=0}^{\infty} = (a, ar, ar^2, \ldots\ldots)$$

call a <u>initial term</u> and

call r <u>common ratio</u>.

- An <u>Arithmetic sequence</u> is of the form

$$(a+nd)_{n=0}^{\infty} = (a, a+d, a+2d, \ldots\ldots)$$

call a <u>initial term</u> and

call d <u>common difference</u>.

Defn

A <u>recurrence relation</u> for (a_n) is an equation expressing a_n as a function of preceding terms.

Ex. $a_n = a_{n-1} + 3 \,,\quad \underbrace{a_0 = 2}_{\text{initial term}}$

$(a_n) = (2, 5, 8, 11, \ldots)$

Ex. $a_n = a_{n-1} - a_{n-2} \,,\quad a_0 = 3,\ a_1 = 5$

$(a_n) = (3, 5, 2, -3, -5, -2, 3, 5, \ldots)$

periodic sequence.

Ex. Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \; F_1 = 1$$

$$(0, 1, 1, 2, 3, 5, 8, 13, \ldots)$$

Ex. Lucas sequence

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \; L_1 = 1$$

$$(2, 1, 3, 4, 7, 11, \ldots)$$

Defn

A solution to a recurrence relation

$$* \quad a_n = F(n, a_{n-1}, \ldots)$$

is a closed formula for a_n (a function of n only).

$$a_n = f(n)$$

that when substituted into both sides of * yields an identity for all n.

Ex. $\begin{cases} a_0 = 1 \\ a_n = n \cdot a_{n-1} \end{cases}$

$(1, 1, 2, 6, 24, 120, \ldots\ldots)$

<u>solution</u> $a_n = n!$

Ex. $\begin{cases} b_0 = 1 \\ b_n = 3 \cdot b_{n-1} \end{cases}$

$(1, 3, 9, 27, 81, \ldots\ldots)$

<u>solution</u> : $a_n = 3^n$

__Ex.__ $c_n = 5c_{n-1} - 6c_{n-2}$ $(n \geq 2)$

__check__ $c_n = 2^n$ yes

$c_n = 3^n$ exercise (yes)

$c_n = 5^n$ no

__$c_n = 2^n$__ ✓

$$RHS = 5c_{n-1} - 6c_{n-2} = 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2}$$

$$= 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} = (5-3) \cdot 2^{n-1}$$

$$= 2 \cdot 2^{n-1} = 2^n = c_n = LHS.$$

__$c_n = 5^n$__ ✗

$$RHS = 5 \cdot c_{n-1} - 6 \cdot c_{n-2} = 5 \cdot 5^{n-1} - 6 \cdot 5^{n-2}$$

$$= 5^n - \frac{6}{25} \cdot 5^2 \cdot 5^{n-2} = 5^n - \frac{6}{25} \cdot 5^n$$

$$= \frac{19}{25} \cdot 5^n \neq 5^n = c_n = LHS.$$

Exercise

- Fibonacci : $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

- Lucas : $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

golden ratio

Hint: use that $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$ are roots of the quadratic:

$$x^2 - x - 1 = 0$$

$$x^2 = x + 1$$

i.e.

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \left(\frac{1+\sqrt{5}}{2}\right) + 1$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = \left(\frac{1-\sqrt{5}}{2}\right) + 1$$

Ex. find a recurrence rel. for

$$\begin{array}{cccccccc} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_n & \cdots \\ 1, & 7, & 25, & 79, & 241, & 727, & 2185, & 6559, \ldots \end{array}$$

diff 6 18 54 162 486 1458 4374

ratio 3 3 3 3 3 3

we see that:

$$a_n - a_{n-1} = 3(a_{n-1} - a_{n-2})$$

$$\therefore \begin{cases} a_n = 4a_{n-1} - 3a_{n-2} \\ a_0 = 1 \\ a_1 = 7 \end{cases}$$

check solution: $a_n = -2 + 3^{n+1}$

Defn
A summation (or series) is the sum of a sequence.

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

Ex $\sum_{j=0}^{6} 2^j = 1+2+4+8+16+32+64$

or $\sum_{k \in S} a_k =$

Ex. $\sum_{k \in \{2,4,6\}} k^2 = 2^2 + 4^2 + 6^2$.

Special formulas

(1) $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ ✓

(2) $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(3) $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(4) $$\sum_{k=0}^{n} ar^k = \begin{cases} a(n+1) & \text{if } r=1 \;✓ \\ a\left(\frac{r^{n+1}-1}{r-1}\right) & \text{if } r\neq 1 \;✓ \end{cases}$$

Proof of (1)

let

$$S = 1+2+3+\cdots+(n-2)+(n-1)+n$$

Then

$$S = n+(n-1)+(n-2)+\cdots+3+2+1$$

Thus

$$S + S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$$

$$\therefore \quad 2S = n(n+1)$$

$$\therefore \quad S = \frac{n(n+1)}{2} \qquad \blacksquare$$

<u>Proof of (4) in case $r \neq 1$</u>

Let

$$S = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n}$$

$$rS = \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n} + ar^{n+1}$$

$$\therefore \quad rS - S = ar^{n+1} - a$$

$$\therefore \quad (r-1)S = a(r^{n+1} - 1)$$

$$\therefore \quad S = a\left(\frac{r^{n+1} - 1}{r - 1}\right). \qquad \blacksquare$$

Ex. $$\sum_{k=5}^{10} 3\cdot 2^k = \sum_{k=0}^{10} 3\cdot 2^k - \sum_{k=0}^{4} 3\cdot 2^k$$

$$= 3\left(\frac{2^{11}-1}{2-1}\right) - 3\left(\frac{2^5-1}{2-1}\right)$$

$$= 3\left(2^{11}-2^5\right)$$

$$= 3(2048-32) = 3\cdot 2016 = \boxed{6048}$$

Recall!

$$\begin{cases} a_n = 4a_{n-1} - 3a_{n-2} \\ a_0 = 1 \\ a_1 = 7 \end{cases}$$

Go back to: $\underbrace{a_n - a_{n-1}}_{b_n} = 3(\underbrace{a_{n-1} - a_{n-2}}_{b_{n-1}})$

so let $b_n = a_n - a_{n-1}$ $(n \geq 1)$

$$\begin{cases} b_n = 3 \cdot b_{n-1} \\ b_0 = 2 \end{cases} \longrightarrow \text{soln: } b_n = 2 \cdot 3^n$$

$\therefore$ $a_n - a_{n-1} = 2 \cdot 3^n$

$$\therefore \begin{cases} a_n = a_{n-1} + 2 \cdot 3^n \\ a_0 = 1 \end{cases}$$

so

$$a_1 = 1 + 2 \cdot 3$$

$$a_2 = 1 + 2 \cdot 3 + 2 \cdot 3^2$$

$$a_3 = 1 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3$$

$$\vdots$$

$$a_n = 1 - 2 \cdot 3^0 + (2 \cdot 3^0 + 2 \cdot 3^1 + \cdots + 2 \cdot 3^n)$$

$$= -1 + 2\left(\frac{3^{n+1} - 1}{3 - 1}\right) = -1 + 3^{n+1} - 1$$

$\therefore$ $\boxed{a_n = -2 + 3^{n+1}}$