CSE 16 7-12-23

Defn

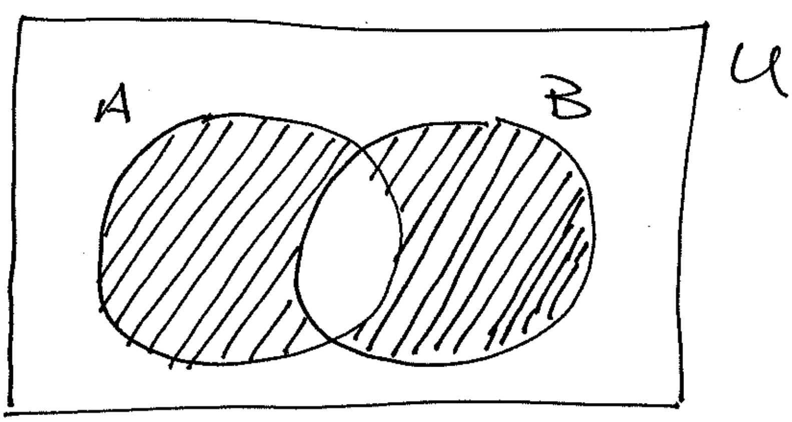
The symmetric difference of A, B

is

$$A \oplus B = \{x \in U \mid x \in A \oplus x \in B\}$$

Exercise: show

- $A \oplus B = (A - B) \cup (B - A)$
- $A \oplus B = (A \cup B) - (A \cap B)$



<u>notation:</u>

Given $A_1, A_2, A_3, \ldots\ldots$

- $A_1 \cup \cdots\cdots \cup A_n = \bigcup_{i=1}^{n} A_i$

- $A_1 \cup A_2 \cup \cdots\cdots = \bigcup_{i=1}^{\infty} A_i$

- $A_1 \cap \cdots\cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

- $A_1 \cap A_2 \cap \cdots\cdots = \bigcap_{i=1}^{\infty} A_i$

2.3 Functions

Defn

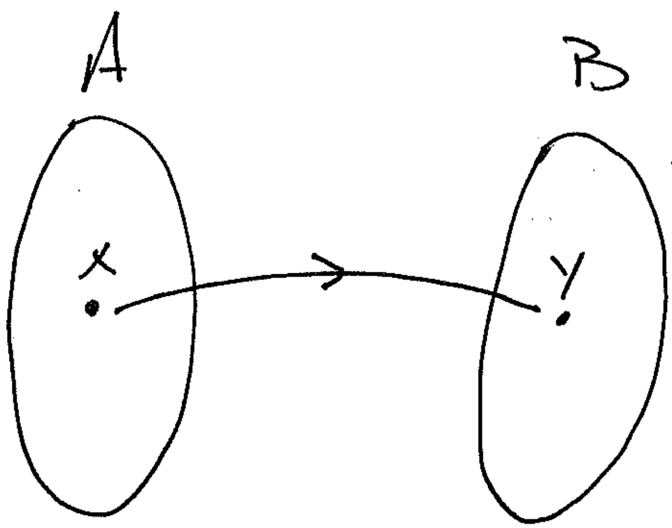
A function (map, mapping, transformation) consists of

(1) A set A called domain

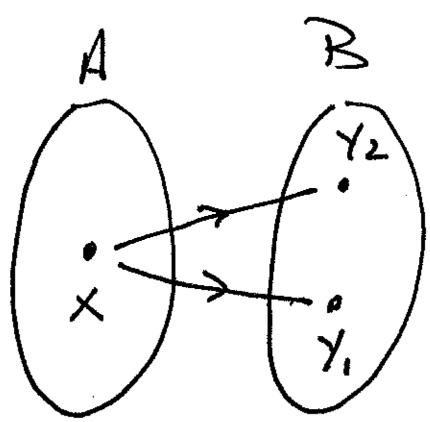
(2) a set B called codomain

(3) a rule f that assigns to each $x \in A$ a unique $y \in B$. write $y = f(x)$. call y the image of x under f, and call x a preimage of y under f.

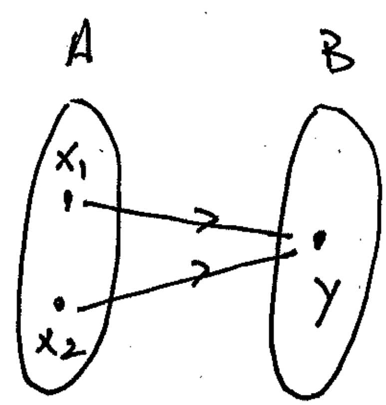
notation: $f : A \to B$



uniqueness



not a function

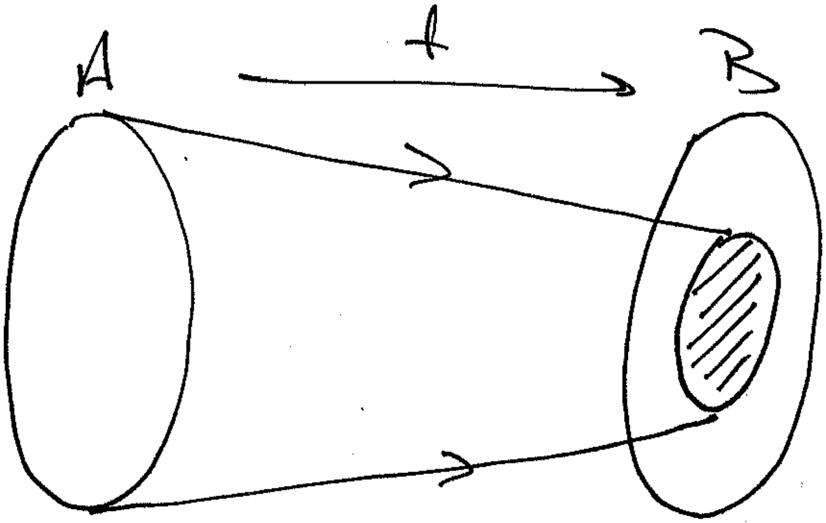


a function

Defn

The <u>range</u> of f is the set

$$\text{range}(f) = \{ y \in B \mid \exists x \in A : y = f(x) \}$$
$$= \{ f(x) \mid x \in A \}$$



<u>Ex.</u> $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

$$\text{range}(f) = \{0, 1, 4, 9, 16, 25, \ldots\ldots\}$$

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$$\text{range}(f) = [0, \infty)$$
$$= \{ y \in \mathbb{R} \mid y \geq 0 \}$$

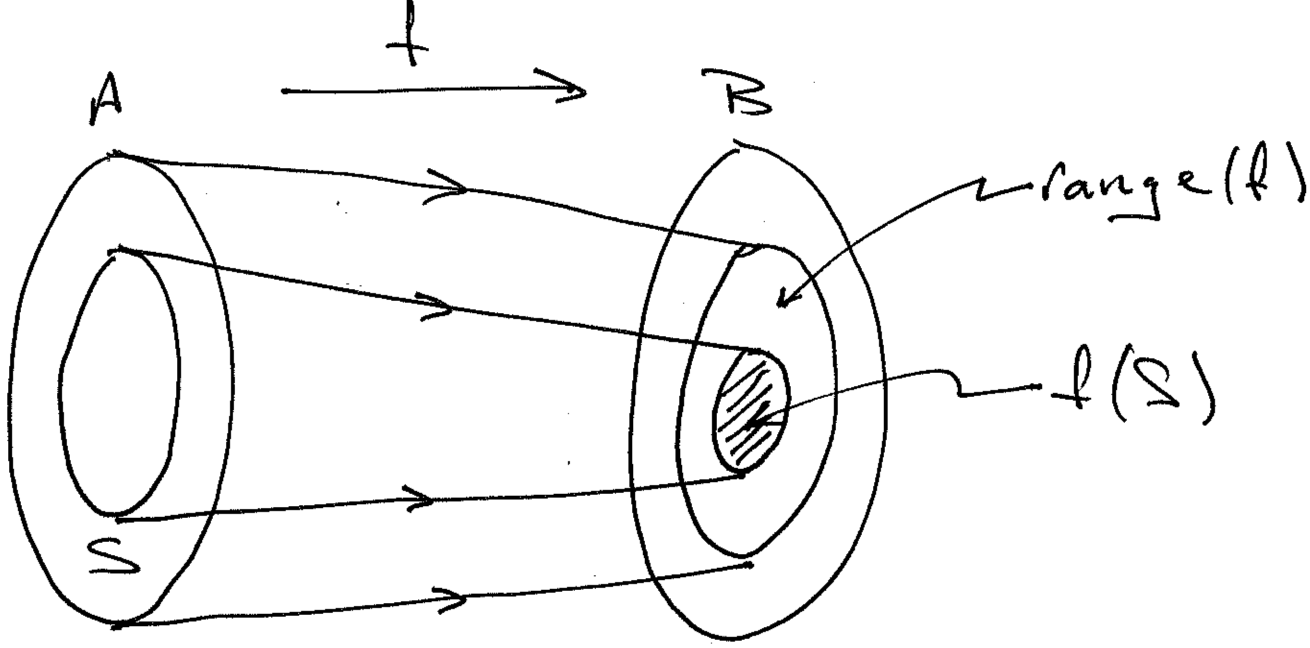
Defn

The image of $S \subseteq A$ under f is

$$f(S) = \{ y \in B \mid \exists x \in S : y = f(x) \}$$
$$= \{ f(x) \mid x \in S \}$$

note: $f(S) \subseteq f(A) = \text{range}(f) \subseteq B$

Picture



Ex. $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

$S = \{-1, 3, 7\}$

$f(S) = \{1, 9, 49\}$

Defn

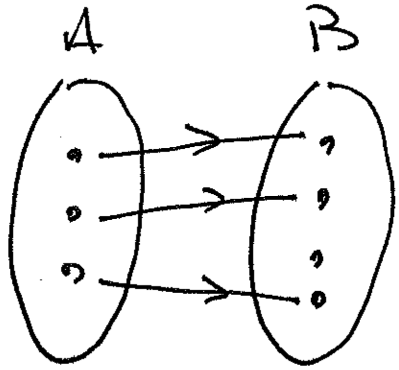
A function $f: A \to B$ is called injective (one-to-one) iff

$$\forall x_1, x_2 \in A : f(x_1) = f(x_2) \to x_1 = x_2$$

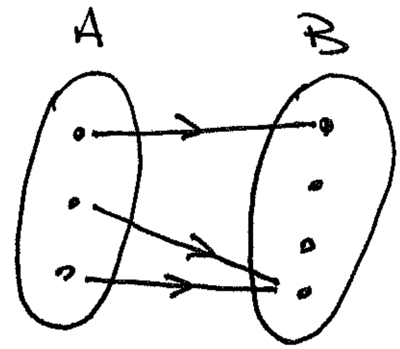
equivalently

$$\forall x_1, x_2 \in A : x_1 \neq x_2 \longrightarrow f(x_1) \neq f(x_2)$$

Picture



injective



not injective

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

not injective: $(-1)^2 = 1 = 1^2$

Ex. $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x + 3$

is injective: Let $x_1, x_2 \in \mathbb{R}$.

Assume $g(x_1) = g(x_2)$. Then

$$2x_1 + 3 = 2x_2 + 3$$

$$\therefore \quad 2x_1 = 2x_2$$

$$\therefore \quad x_1 = x.$$

Defn

We say $f: \mathbb{R} \to \mathbb{R}$ is

- strictly increasing: $x < y \to f(x) < f(y)$
- strictly decreasing: $x < y \to f(x) > f(y)$

Theorem

if $f: \mathbb{R} \to \mathbb{R}$ is either strictly inc. or strictly dec., then f is injective.

Ex. $f: \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = \frac{1}{x}$

is injective

Defn

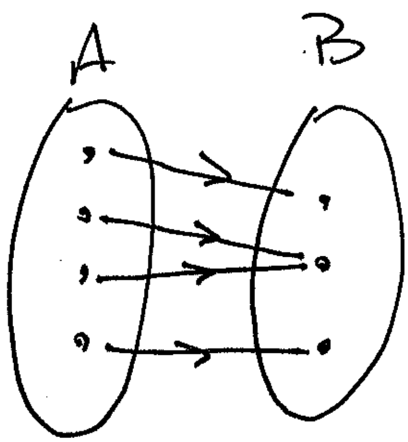
A function $f: A \to B$ is called **surjective** (**onto**) iff

$$\text{range}(f) = B.$$

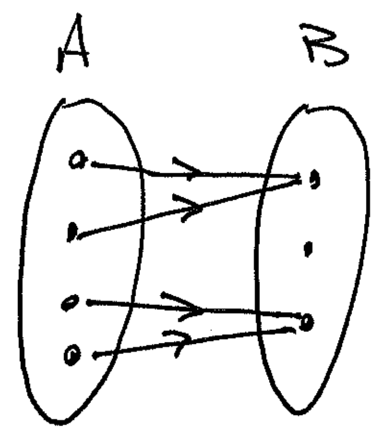
equivalently

$$\forall y \in B \; \exists x \in A : y = f(x)$$

Pictures



surjective



not surjective

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

not surjective: $\nexists\, x$ st. $x^2 = -1$.

Ex. $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x + 3$

is surjective: Let $y \in \mathbb{R}$.

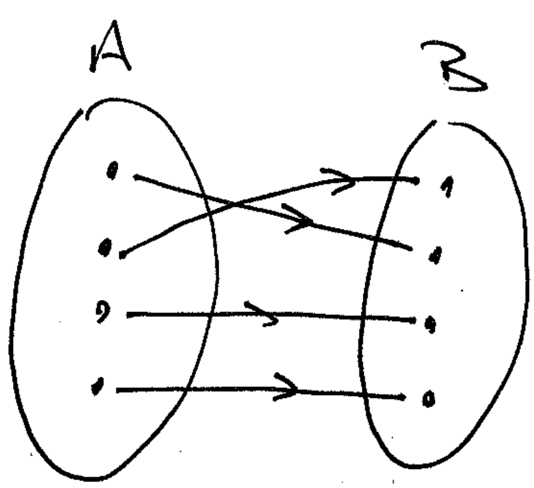
Let $x = \frac{y-3}{2}$. Then

$$g(x) = g\left(\frac{y-3}{2}\right) = \cancel{2}\left(\frac{y-3}{\cancel{2}}\right) + 3$$

$$= (y-3) + 3 = y.$$

Defn

A function $f: A \to B$ is called <u>bijective</u> iff it is both injective and surjective



<u>Ex.</u> $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$ is bijective.

Defn

The <u>composition</u> of $f: A \to B$, $g: B \to C$ is the function

$$g \circ f : A \to C$$

given by

$$g \circ f(x) = g(f(x))$$

for any $x \in A$.

<u>Ex.</u>

let $A = B = C = \mathbb{R}$, $f(x) = x^2$, $g(x) = 2x+3$.

$$g \circ f(x) = g(f(x)) = g(x^2) = 2x^2 + 3$$

and

$$f \circ g(x) = f(g(x)) = f(2x+3) = (2x+3)^2$$
$$= 4x^2 + 12x + 9$$

<u>note</u>: $f \circ g \neq g \circ f$ in general.

Theorem

Let $f: A \to B$, $g: B \to C$. Then

(a) f, g injective $\to$ $g \circ f$ injective

(b) f, g surjective $\to$ $g \circ f$ surjective

(c) f, g bijective $\to$ $g \circ f$ bijective

Proof of (a)

Let $x_1, x_2 \in A$ and suppose that $g \circ f(x_1) = g \circ f(x_2)$. Then

$$g(f(x_1)) = g(f(x_2))$$

$\therefore$ $f(x_1) = f(x)$ (since g inj.)

$\therefore$ $x_1 = x_2$ (since f inj)

$\blacksquare$

Exercise

Prove b and c.

Exercise

Prove that composition is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$ for appropriate h, g, f.

Invertibility

Define for sets A, B

$$i_A : A \to A \quad \text{by} \quad i_A(x) = x$$
$$i_B : B \to B \quad \text{by} \quad i_B(y) = y$$

These are called identity functions.

<u>note</u>: for any $f: A \to B$

$$f \circ i_A = f = i_B \circ f$$

<u>Defn</u>

A function $f: A \to B$ is called <u>invertible</u> iff there exists $g: B \to A$ such that

$$f \circ g = i_B \text{ and } g \circ f = i_A$$

g is called an <u>inverse function</u> for f.

Theorem

if f is invertible, then its inverse g is unique.

Proof

Suppose there exist $g_1 : B \to A$ and $g_2 : B \to A$ satisfying

- $f \circ g_1 = i_B$ and $g_1 \circ f = i_A$
- $f \circ g_2 = i_B$ and $g_2 \circ f = i_A$

Then

$$g_1 = i_A \circ g_1$$

$$= (g_2 \circ f) \circ g_1$$

$$= g_2 \circ (f \circ g_1) \quad \text{(assoc.)}$$

$$= g_2 \circ i_B = g_2$$

∎

Now we can speak of <u>the</u> inverse function for $f : A \to B$ which is invertible. write

$$f^{-1} : B \to A$$

for g.

<u>Theorem</u>

$f : A \to B$ is invertible iff $\overset{\Longrightarrow}{\underset{\Longleftarrow}{}}$ f is bijective.

Proof

($\Longrightarrow$)

Assume f is invertible, so $f^{-1}: B \to A$ exists. Then

- f is injective

Let $x_1, x_2 \in A$ and assume $f(x_1) = f(x_2)$.

$\therefore \quad f^{-1}(f(x_1)) = f^{-1}(f(x_2))$

$\therefore \quad f^{-1} \circ f(x_1) = f^{-1} \circ f(x_2)$

$\therefore \quad i_A(x_1) = i_A(x_2)$

$\therefore \quad x_1 = x_2$. $\square$

- f is surjective

Let $y \in B$, and set $x = f^{-1}(y)$.

$$\therefore \; f(x) = f(f^{-1}(y))$$
$$= f \circ f^{-1}(y)$$
$$= i_B(y)$$
$$= y. \qquad \square$$

Thus f is bijective.

($\Longleftarrow$)

Assume f is bijective. Define
$g: B \to A$ by:

for each $y \in B$

$$g(y) = \text{the unique } x \in A \text{ s.t. } y = f(x)$$

Note: such an x exists since f is surjective, it is unique since f is inj.

Then we have

$$f(g(y)) = y \quad \text{for all } y \in B$$

and

$$g(f(x)) = x \quad \text{for all } x \in A$$

Thus

$$f \circ g = i_B \quad \text{and} \quad g \circ f = i_A$$

$\therefore\ g = f^{-1}$ and f is invertible