cse 16 6-30-23

Recall:

- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

	True when	false when
$\forall x\, P(x)$	true for all $x \in U$	false for at least one $x \in U$
$\exists x\, P(x)$	true for at least one $x \in U$	false for all $x \in U$

<u>Defn</u>

A variable x is called

- <u>Bound</u> : if it has been either Quantified ($\forall x, \exists x$), or substituted for.
- <u>Free</u> : if it is not bound.

<u>Ex.</u>

	x	y	z
$P(x,y) \vee Q(z)$	<u>f</u>ree	f	f
$\forall x\, P(x,y) \vee Q(z)$	<u>b</u>ound	f	f
$\exists x \forall z\, P(x,y) \vee Q(z)$	b	f	b
$\exists z\, P(6,8) \vee Q(z)$	b	b	b

Translation

Ex. 'All horses in CA are blue'

- let $U = \{\text{horses in CA}\}$

  $B(x) =$ 'x is blue'

$$\forall x\, B(x)$$

- let $U = \{\text{horses}\}$, $C(x) =$ 'x lives in CA'

$$\forall x\, (C(x) \rightarrow B(x))$$

- let $U = \{\text{living things}\}$, $H(x) =$ 'x is a horse'

$$\forall x\, ((H(x) \wedge C(x)) \rightarrow B(x))$$

note: this is wrong

$$\forall x \, (H(x) \wedge C(x) \wedge B(x))$$

1.5 Nested Quantifiers

what does $\equiv$ mean for predicates & quantifiers

Defn
Let A, B be expressions with predicates and quntifiers, where all variables are bound. Then $A \equiv B$ iff A and B have same truth value for all prop. functions and all universes.

Ex. $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

Ex. $\neg \forall x\, P(x) \not\equiv \forall x\, \neg P(x)$

why? let $P(x) =$ '$x>0$', $U = \mathbb{R}$

Then

$\neg \forall x\, P(x)$ is true

$\forall x\, \neg P(x)$ is false.

let $P(x,y)$ be a Prop. fcn. with

$\mathcal{U}_x = \mathcal{U}_y = \mathcal{U}$.

(1) $\forall x \forall y \; P(x,y) \equiv \forall y \forall x \; P(x,y)$ ✓

(2) $\exists x \exists y \; P(x,y) \equiv \exists y \exists x \; P(x,y)$ ✓ exerc.

(3) $\forall x \exists y \; P(x,y) \not\equiv \exists y \forall x \; P(x,y)$

• do (1) with $\mathcal{U} = \{1,2\}$:

$$
\begin{aligned}
\forall x \forall y \, P(x,y) &\equiv \forall x \left(P(x,1) \wedge P(x,2) \right) \\
&\equiv \left(P(1,1) \wedge P(1,2) \right) \wedge \left(P(2,1) \wedge P(2,2) \right) \\
&\equiv \left(P(1,1) \wedge P(2,1) \right) \wedge \left(P(1,2) \wedge P(2,2) \right) \\
&\equiv \forall y \left(P(1,y) \wedge P(2,y) \right) \\
&\equiv \forall y \forall x \; P(x,y)
\end{aligned}
$$
✓

- why $\not\equiv$ in (3) ?

let $P(x,y) = $ '$x<y$', $U=\mathbb{R}$

Then

$\forall x \exists y P(x,y)$ is true

$\exists y \forall x P(x,y)$ is false

<u>more Translation</u>

let $L(x,y) = $ 'x loves y'

$U = \{\text{people}\}$

(1) 'everybody loves somebody'

$$\forall x \exists y L(x,y)$$

(2) 'somebody loves everybody'

$$\exists x \, \forall y \; L(x,y)$$

(3) 'nobody loves everybody'

$$\neg \exists x \, \forall y \; L(x,y)$$

$$\equiv \forall x \, \neg \forall y \; L(x,y)$$

$$\equiv \forall x \, \exists y \; \neg L(x,y)$$

(4) 'there is exactly one person whom everyone loves'

$$\exists y \left(\forall x\, L(x,y) \wedge \forall z \left(z \neq y \rightarrow \exists w\, \neg L(w,z) \right) \right)$$

(5) 'there is a person who loves everyone except himself'

$$\exists x \left(\neg L(x,x) \wedge \forall y \left(y \neq x \rightarrow L(x,y) \right) \right)$$

Scope of Quantifiers

(1) $\forall x \underbrace{(P(x) \rightarrow Q(x))} \not\equiv \forall x \underbrace{P(x)} \rightarrow \forall x \underbrace{Q(x)}$

(2) $\exists x (P(x) \rightarrow Q(x)) \not\equiv \exists x P(x) \rightarrow \exists x Q(x)$

(3) $\exists x (P(x) \wedge Q(x)) \not\equiv \exists x P(x) \wedge \exists x Q(x)$

- (1) & (2) are $\not\equiv$ since:

  $U = \{\text{people}\}$

  $P(x) =$ 'x has green eyes'

  $Q(x) =$ 'x is 50 ft. tall'

- (3) is $\not\equiv$ since

  $U = \mathbb{R}$

  $P(x) = $ '$x > 0$'

  $Q(x) = $ '$x < 0$'

Exercise

(4) $\forall x\,(P(x) \vee Q(x))$? $\forall x P(x) \vee \forall x Q(x)$

(5) $\exists x\,(P(x) \vee Q(x))$? $\exists x P(x) \vee \exists Q(x)$

(6) $\forall x\,(P(x) \wedge Q(x))$? $\forall x P(x) \wedge \forall x Q(x)$

Hint: try $U = \{1, 2\}$.

Skip 1.6

more from 1.1, 1.2

see 1.1 # 48, 49, 50

Ex p = 'this statement is false'

$p=1 \longrightarrow p=0$ $\times$

$p=0 \longrightarrow p=1$ $\times$

Thus p is <u>not</u> a proposition.

Ex. p (1) 'stmt 2 is false' : $p=\neg q = \neg\neg p$
q (2) 'stmt 1 is false' : $q=\neg p=\neg\neg q$

p	q	
0	0	$\times$
0	1	possible
1	0	possible
1	1	$\times$

Ex.

① 'exactly 1 stmt in this list is false'

② ' " 2 " " " " " " '

③ ' " 3 " " " " " " '

①	②	③	
0	0	0	X
0	0	1	X
0	1	0	Possible
0	1	1	X
1	0	0	X
1	0	1	X
1	1	0	X
1	1	1	X

exercise