CSE 16 6-29-23

Defn

Two (compound) propositions A, B are said to be logically equivalent iff $A \leftrightarrow B$ is a tautology.

notation: $A \equiv B$ (or $A \Leftrightarrow B$)

In other words, $A \equiv B$ iff A and B have same truth table.

Ex $p \rightarrow q \equiv \neg p \vee q$ ✓

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

Ex. $p \to q \equiv \neg q \to \neg p$

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Ex. $p \to q \not\equiv q \to p$

p	q	$p \to q$	$q \to p$
0	0	1	1
0	1	1	0
1	0	0	1
1	1	1	1

Exercise:

Prove all equivalences on tables 6, 7, 8 in sec. 1.3

Table 6

- Identity $p \land T \equiv p$
  $p \lor F \equiv p$

- Domination $p \lor T \equiv T$
  $p \land F \equiv F$

- Idempotent $p \lor p \equiv p$
  $p \land p \equiv p$

- Double Negation $\neg(\neg p) \equiv p$

- Commutative $\quad p \vee q \equiv q \vee p$

  $p \wedge q \equiv q \wedge p$

- Associative $\quad (p \vee q) \vee r \equiv p \vee (q \vee r)$

  $\wedge \quad \wedge \qquad \wedge \quad \wedge$

- Distributive $\quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

  $\wedge \quad \vee \qquad \wedge \quad \vee \quad \wedge$

- DeMorgan $\quad \neg (p \wedge q) \equiv \neg p \vee \neg q$ ✓

  $\vee \qquad \wedge$

- Absorbtion $\quad p \vee (p \wedge q) \equiv p$

  $\wedge \quad \vee$

- Negation $\quad p \vee \neg p \equiv T$

  $p \wedge \neg p \equiv F$

Table 7

- $p \to q \equiv \neg p \vee q$ ✓
- $p \to q \equiv \neg q \to \neg p$ ✓
- $p \vee q \equiv \neg p \to q$ ✓
- $p \wedge q \equiv \neg(p \to \neg q)$

$\vdots$

Proof of 1st DeMorgan

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

Algebraic Proofs of identities

Ex show $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

Proof **Reason**

$$
\begin{aligned}
(p \vee q) \to r &\equiv \neg(p \vee q) \vee r && \text{T7 \#1} \\
&\equiv (\neg p \wedge \neg q) \vee r && \text{DeMorgan} \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \text{comm, Dist} \\
&\equiv (p \to r) \wedge (q \to r) && \text{T7 \#1}
\end{aligned}
$$

$\blacksquare$

Ex. show $(p \wedge q) \to (p \to q)$ is a tautology

Proof

$$(p \wedge q) \to (p \to q) \equiv \neg(p \wedge q) \vee (\neg p \vee q) \quad \text{T7.\#1}$$

$$\equiv (\neg p \vee \neg q) \vee (\neg p \vee q) \quad \text{DeMorgan}$$

$$\equiv (\neg p \vee \neg p) \vee (\neg q \vee q) \quad \text{comm, assoc.}$$

$$\equiv \neg p \vee T \quad \text{Idem, Neg.}$$

$$\equiv T \quad \text{Domination}$$

∎

Exercise show $\neg(p \to q) \to p$ is a tautology

Exercise Determine whether

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

is satisfiable. (ans. yes)

1.4 Predicates and Quantifiers

So far we cannot say something like: '$2x=3$ has a solution'

Defn

A Propositional Function (or Predicate) is a statement $P(x)$ that becomes a proposition when a value of x is specified.

Ex. $P(x) =$ '$x<7$' $\quad U = \mathbb{R}$

Ex. $P(x) =$ 'x is a sunny day' $\quad U = \{\text{days}\}$

Ex. $Q(x,y) =$ '$x+y=10$' $\quad U_x = U_y = \mathbb{R}$

The <u>Universe of Discourse</u> (<u>Domain</u>) of a Prop. Fcn. is the set of possible values for its variable(s).

write: $\mathcal{U}$

Another way to obtain a proposition from a predicate: <u>Quantification</u>

<u>Defn</u>

The <u>Universal Quantification</u> of $P(x)$ is the proposition

'$P(x)$ is true for all $x \in \mathcal{U}$'

<u>notation</u>: $\forall x \, P(x)$

<u>read</u>: 'for all x, $P(x)$'

<u>Defn</u>

The <u>Existential Quantification</u> of $P(x)$

is

'$P(x)$ is true for at least one $x \in \mathcal{U}$'

<u>notation</u>: $\exists x\, P(x)$

<u>read</u>: 'there exists an x such that $P(x)$'.

<u>Ex.</u> $P(x) =$ '$x^2 \geq 0$'. $\mathcal{U} = \mathbb{R}$

<u>Ex.</u> $Q(x) =$ '$x < 50$'. $\mathcal{U} = \mathbb{R}$

<u>Ex.</u> $R(x) =$ '$2x = 3$'. $\mathcal{U} = \mathbb{R}$

Try replacing $\mathbb{R}$ by $\mathbb{Z} = \{\text{integers}\}$

consider a finite universe

$$U = \{1, 2\}.$$

Then

$$\forall x\, P(x) \equiv P(1) \wedge P(2)$$

$$\exists x\, P(x) \equiv P(1) \vee P(2)$$

so

$$\neg \forall x\, P(x) \equiv \neg \big(P(1) \wedge P(2)\big)$$

$$\equiv \neg P(1) \vee \neg P(2)$$

$$\equiv \exists x\, \neg P(x)$$

Also

$$\neg \exists_x P(x) \equiv \neg \left(P(1) \vee P(2)\right)$$

$$\equiv \neg P(1) \wedge \neg P(2)$$

$$\equiv \forall_x \neg P(x)$$