CSE 16
Summer 2022
Quiz 6 **Solutions**

1. (20 Points) Use induction to prove

$$\forall n \geq 1: \quad \sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof:
I. If $n = 1$, the above statement is $1/(1+1)! = 1 - 1/(1+1)!$, i.e. $1/2 = 1/2$, which is true.

II. Let $n \geq 1$ be arbitrary. Assume

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

We must show

$$\sum_{k=1}^{n+1} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+2)!}.$$

Observe

$$\sum_{k=1}^{n+1} \frac{k}{(k+1)!} = \left(\sum_{k=1}^{n} \frac{k}{(k+1)!}\right) + \frac{n+1}{(n+2)!}$$

$$= \left(1 - \frac{1}{(n+1)!}\right) + \frac{n+1}{(n+2)!} \qquad \text{(by the induction hypothesis)}$$

$$= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!}$$

$$= 1 - \left(\frac{(n+2)-(n+1)}{(n+2)!}\right)$$

$$= 1 - \frac{1}{(n+2)!}.$$

The result follows for all $n \geq 1$ by the 1st PMI. ■

2. (20 Points) Let x, y and n denote integers. Use induction to prove that

$$\forall n \geq 5: \ \exists x, y \geq 0 \ \text{ such that } \ n = 2x + 3y$$

(Hint: use strong induction form IId, with two base cases: $n = 5$ and $n = 6$.)

Proof:

I. We have $5 = 2 \cdot 1 + 3 \cdot 1$ and $6 = 2 \cdot 0 + 3 \cdot 2$, showing that the result is true when $n = 5$ and when $n = 6$, respectively.

II. Let $n > 6$ be chosen arbitrarily. Assume for all k in the range $5 \leq k < n$ that

$$\exists x_k, y_k \geq 0 \ \text{ such that } \ k = 2x_k + 3y_k.$$

We must show

$$\exists x, y \geq 0 \ \text{ such that } \ n = 2x + 3y.$$

Observe $n > 6 \ \Rightarrow \ n \geq 7 \ \Rightarrow \ n - 2 \geq 5$. The induction hypothesis with $k = n - 2$, gives us integers $x', y' \geq 0$ such that $n - 2 = 2x' + 3y'$. Let $x = x' + 1$ and $y = y'$. Then

$$\begin{aligned} n &= (n-2) + 2 \\ &= (2x' + 3y') + 2 \\ &= 2(x' + 1) + 3y' \\ &= 2x + 3y, \end{aligned}$$

as required. The result follows for all $n \geq 5$ by the 2nd PMI. ∎

3. (10 Points) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and define a relation $R \subseteq A \times A$ by

$$R = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,4), (5,6), (5,7), (6,6), (6,7)\}$$

Determine (**yes** or **no**) whether R has the following properties (no justification is required).

Solution:

Reflexive? **no**
Symmetric? **no**
Transitive? **yes**
Anti-symmetric? **yes**

∎