CSE 16
Summer 2022
Quiz 5 **Solutions**

1. (15 Points) Let a, $b \in \mathbb{Z}$ and $m \in \mathbb{N}$. Prove that if $a \equiv b(\text{mod}\, m)$, then $a^3 \equiv b^3(\text{mod}\, m)$.

   Proof:
   We have: $a \equiv_m b \implies m|(a-b) \implies (a-b) = mk$, for some $k \in \mathbb{Z}$. Therefore

   $$a^3 - b^3 = (a-b)(a^2 + ab + b^2) = mk(a^2 + ab + b^2),$$

   hence $m|(a^3 - b^3)$, and thus $a^3 \equiv_m b^3$, as required. ■

2. (15 Points) Let a, b, $c \in \mathbb{Z}$. Prove that if $a|bc$ and $\gcd(a, b) = 1$, then $a|c$. [Hint: use that fact that $\gcd(a, b) = an + bm$ for some n, $m \in \mathbb{Z}$.]

   Proof:
   Following the hint we have

   $$\gcd(a, b) = 1 \implies an + bm = 1 \quad \text{(for some } n, m \in \mathbb{Z}.)$$

   Also

   $$a|bc \implies bc = ak \quad \text{(for some } k \in \mathbb{Z}.)$$

   Multipling the first equation by c gives

   $$\begin{aligned} & acn + bcm = c, \\ \therefore\quad & acn + akm = c, \\ \therefore\quad & a(cn + km) = c, \end{aligned}$$

   whence $a|c$, as required. ■

3. (20 Points) Use induction to prove that for all $n \geq 1$:

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof:

I. If $n = 1$, the above formula says $\sum_{k=1}^{1} k(k+1) = \frac{1\cdot(1+1)\cdot(1+2)}{3}$, i.e. $1 \cdot (1+1) = \frac{1\cdot(1+1)\cdot(1+2)}{3}$, i.e. $1 \cdot 2 = \frac{1\cdot 2\cdot 3}{3}$, i.e. $2 = 2$, which is true.

II. Let $n \geq 1$ be chosen arbitrarily. Assume (for this n) that

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

We must show that

$$\sum_{k=1}^{n+1} k(k+1) = \frac{(n+1)(n+2)(n+3)}{3}.$$

Observe

$$\begin{aligned}
\sum_{k=1}^{n+1} k(k+1) &= \left(\sum_{k=1}^{n} k(k+1)\right) + (n+1)\big((n+1)+1\big) \\
&= \left(\frac{n(n+1)(n+2)}{3}\right) + (n+1)(n+2) \qquad \text{(by the induction hypothesis)} \\
&= (n+1)(n+2)\left(\frac{n}{3}+1\right) \\
&= \frac{(n+1)(n+2)(n+3)}{3}.
\end{aligned}$$

The result follows for all $n \geq 1$ by the Principle of Mathematical Induction. ∎