CSE 16
Summer 2022
Quiz 4 **Solutions**

1. (10 Points) Let $a, b, c, m \in \mathbb{Z}$, with $m \geq 1$. Prove that if $m|a$ and $m|b$, then $m|(a+b)$.

   Proof:
   By the definition of divisibility, we have $a = k_1 m$ and $b = k_2 m$ for some $k_1, k_2 \in \mathbb{Z}$. Therefore

   $$a + b = k_1 m + k_2 m = (k_1 + k_2)m,$$

   and hence $m|(a+b)$. ■

2. (20 Points) Prove that $\sqrt{3}$ is irrational. [Hint: you may use without proof that $3|n$ if and only if $3|n^2$.]

   Proof:
   Assume, to get a contradiction, that $\sqrt{3}$ is rational, and hence $\sqrt{3} = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. We may assume that a and b are relatively prime, since if they have any prime factors in common, they can be canceled from the fraction a/b. But then

   $$\begin{aligned}
   \sqrt{3} = a/b \;&\rightarrow\; 3 = a^2/b^2 \\
   &\rightarrow\; 3b^2 = a^2 \\
   &\rightarrow\; 3|a^2 \\
   &\rightarrow\; 3|a && \text{(by the above hint)} \\
   &\rightarrow\; a = 3k, \text{ for some } k \in \mathbb{Z} \\
   &\rightarrow\; 3b^2 = a^2 = (3k)^2 = 9k^2 \\
   &\rightarrow\; b^2 = 3k^2 \\
   &\rightarrow\; 3|b^2 \\
   &\rightarrow\; 3|b && \text{(again by the hint).}
   \end{aligned}$$

   Therefore a and b are both divisible by 3, contradicting that they have no common factors. This contradiction shows that our assumption was false, and hence $\sqrt{3}$ is irrational. ■

3. (20 Points) Let $0 \le s \le r \le k \le n$. Give a combinatorial proof of the following identity.

$$\binom{n}{k}\binom{k}{r}\binom{r}{s} = \binom{n}{s}\binom{n-s}{r-s}\binom{n-r}{k-r}$$

[Hint: count the number of triples (A, B, C) such that $A \subseteq B \subseteq C \subseteq T$, where $|A| = s$, $|B| = r$, $|C| = k$ and $|T| = n$ in two different ways.]

Note: you may attempt an algebraic proof for reduced credit of at most 12/20 points.

Combinatorial Proof:
Following the hint, let T be an n-element set. We count the set of triples (A, B, C) described above in two ways.

I. Choose the subsets in the order: C, B, A.

(1) $C \subseteq T$ with $|C| = k$ can be chosen in $\binom{n}{k}$ ways.
(2) $B \subseteq C$ with $|B| = r$ can be chosen in $\binom{k}{r}$ ways.
(3) $A \subseteq B$ with $|A| = s$ can be chosen in $\binom{r}{s}$ ways.

By the multiplication principle, the number of such triples is $\binom{n}{k}\binom{k}{r}\binom{r}{s}$.

II. Choose the subsets in the order: A, B, C.

(1) $A \subseteq T$ with $|A| = s$ can be chosen in $\binom{n}{s}$ ways.
(2) From the $n - s$ elements not in A, choose $r - s$ elements and add them to A to form B with $|B| = s + (r - s) = r$. This can be done in $\binom{n-s}{r-s}$ ways. Observe that now $A \subseteq B \subseteq T$.
(3) From the $n - r$ elements not in B, choose $k - r$ elements and add them to B to form C with $|C| = r + (k - r) = k$. This can be done in $\binom{n-r}{k-r}$ ways. At this point $A \subseteq B \subseteq C \subseteq T$.

By the multiplication principle, the number of such triples is $\binom{n}{s}\binom{n-s}{r-s}\binom{n-r}{k-r}$. ∎

Algebraic Proof:

$$\text{RHS} = \frac{n!}{s!\,(n-s)!} \cdot \frac{(n-s)!}{(r-s)!\,\big((n-s)-(r-s)\big)!} \cdot \frac{(n-r)!}{(k-r)!\,\big((n-r)-(k-r)\big)!}$$

$$= \frac{n!\,\cancel{(n-s)!}\,\cancel{(n-r)!}}{s!\,\cancel{(n-s)!}\,(r-s)!\,\cancel{(n-r)!}\,(k-r)!\,(n-k)!}$$

$$= \frac{n!\,k!\,r!}{k!\,(n-k)!\,r!\,(k-r)!\,s!\,(r-s)!}$$

$$= \frac{n!}{k!\,(n-k)!} \cdot \frac{k!}{r!\,(k-r)!} \cdot \frac{r!}{s!\,(r-s)!} = \text{LHS}.$$

∎