CSE 16
Summer 2022
Quiz 3 **Solutions**

1. (20 Points) Determine the coefficient of x^{34} in the full expansion of $\left(x^2 - \frac{2}{x}\right)^{20}$. Also determine the coefficient of x^{-17} in the same expansion.

   Solution:
   By the binomial theorem

$$\left(x^2 - \frac{2}{x}\right)^{20} = \sum_{k=0}^{20} \binom{20}{k} \cdot (x^2)^{20-k} \cdot \left(-\frac{2}{x}\right)^k$$

$$= \sum_{k=0}^{20} (-1)^k \cdot 2^k \cdot \binom{20}{k} x^{40-2k-k}$$

$$= \sum_{k=0}^{20} (-1)^k \cdot 2^k \cdot \binom{20}{k} x^{40-3k}$$

   The monomial x^{34} occurs when $40 - 3k = 34 \;\Rightarrow\; 3k = 6 \;\Rightarrow\; \boldsymbol{k = 2}$. Hence

$$(\text{coeficient of } x^{34}) = (-1)^2 \cdot 2^2 \cdot \binom{20}{2} = 4 \cdot 190 = \mathbf{760}.$$

   The monomial x^{-17} occurs when $40 - 3k = -17 \;\Rightarrow\; 3k = 57 \;\Rightarrow\; \boldsymbol{k = 19}$. Therefore

$$(\text{coeficient of } x^{-17}) = (-1)^{19} \cdot 2^{19} \cdot \binom{20}{19} = (-1) \cdot 2^{19} \cdot 20 = \mathbf{-10,485,760}$$

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2. (15 Points) Determine the number of integer solutions (x, y, z, w) to the equation

$$x + y + z + w = 40$$

   that satisfy $x \geq 0$, $y \geq 0$, $z \geq 6$ and $w \geq 4$.

   Solution:
   Let $u = z - 6$ and $v = w - 4$. Then $z \geq 6 \;\Rightarrow\; u \geq 0$ and $w \geq 4 \;\Rightarrow\; v \geq 0$. Therefore the above equation is equivalent to $x + y + (z - 6) + (w - 4) = 40 - 6 - 4$, which is equivalent to

$$x + y + u + v = 30$$

   where $x \geq 0$, $y \geq 0$, $u \geq 0$ and $v \geq 0$. The number of solutions to both equations is therefore the number of 30-element multisets based on a 4-element set, which is $\binom{30+4-1}{30} = \binom{33}{30} = \mathbf{5,456}$.

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3. (15 Points) A six-sided die has faces labeled $\{1, 2, 3, 4, 5, 6\}$. What is the fewest number of rolls necessary to guarantee that at least 20 of the rolls result in the same number on the top face?

Solution:
Suppose the die is rolled n times. Each roll results in one of 6 possible outcomes $\{1, 2, 3, 4, 5, 6\}$. By the Pigeonhole Principle, when n objects (rolls) are placed in 6 boxes (numbers in $\{1, 2, 3, 4, 5, 6\}$), at least one box $\{1, 2, 3, 4, 5, 6\}$ contains at least $\left\lceil \frac{n}{6} \right\rceil$ objects (rolls). Thus we seek the smallest number n such that

$$\left\lceil \frac{n}{6} \right\rceil = 20 \;\Rightarrow\; 19 < \frac{n}{6} \leq 20 \;\Rightarrow\; 114 < n \leq 120.$$

The smallest such number is therefore $n = 114 + 1 = \mathbf{115}$. ■