CSE 16
Summer 2022
Quiz 2 **Solutions**

1. (15 Points) Let $K(x, y)$ be the statement "x knows y", where the universe of discourse for both x and y is the set of all people. (Assume it is possible for x to know y while y does not know x, and hence $K(x, y)$ and $K(y, x)$ are different statements.) Translate the sentence "everybody knows somebody and somebody knows everybody" into predicate logic.

   Solution:

   $$\big(\forall x\ \exists y\ K(x, y)\big) \wedge \big(\ \exists z\ \forall w\ K(z, w)\big)$$

   Alternate Solution:

   $$\big(\forall x\ \exists y\ K(x, y)\big) \wedge \big(\ \exists x\ \forall y\ K(x, y)\big)$$

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2. (15 Points) A banking system requires Personal Identification Numbers (PINs) that satisfy the following constraints.

   (1) Must consist of only decimal digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
   (2) Must have length at least 5, and at most 8.
   (3) Must not be the customer's birth date. (Here, a birth date would be encoded as a 6 digit number, so for instance May 3, 1998 would be 050398)

   How many different valid PINs are there?

   Solution:
   We partition the PINs satisfying (1) and (2) into 4 sets. By the multiplication principle we have

   length 5: # of PINs $= 10^5$
   length 6: # of PINs $= 10^6$
   length 7: # of PINs $= 10^7$
   length 8: # of PINs $= 10^8$

   There is only one PIN violating condition (3), so by the addition and subtraction principles, the number of valid PINs is: $10^5 + 10^6 + 10^7 + 10^8 - 1 = \mathbf{111{,}099{,}999}$. ■

3. (20 Points) Let $0 \le k \le n$. Show that $P(n,k) = k \cdot P(n-1,k-1) + P(n-1,k)$. You may use an algebraic or combinatorial proof.

Algebraic Proof:

$$\begin{aligned}
\text{RHS} &= k \cdot P(n-1,k-1) + P(n-1,k) \\
&= k \cdot \frac{(n-1)!}{\big((n-1)-(k-1)\big)!} + \frac{(n-1)!}{\big((n-1)-k\big)!} \\
&= \frac{k \cdot (n-1)!}{(n-k)!} + \frac{(n-k) \cdot (n-1)!}{(n-k) \cdot (n-k-1)!} \\
&= \frac{k \cdot (n-1)!}{(n-k)!} + \frac{(n-k) \cdot (n-1)!}{(n-k)!} \\
&= \frac{(k+n-k) \cdot (n-1)!}{(n-k)!} \\
&= \frac{n \cdot (n-1)!}{(n-k)!} \\
&= \frac{n!}{(n-k)!} = P(n,k) = \text{LHS}
\end{aligned}$$

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Combinatorial Proof:
The left hand side counts the number of k-permutations of an n-element set, or equivalently, the number of strings of length k from an alphabet of size n with no repeated symbols. We count the set another way by partitioning it into two disjoint classes of such strings.

Let the alphabet be $A = \{1, 2, 3, \ldots, (n-1), n\}$, so that $A - \{n\} = \{1, 2, 3, \ldots, (n-1)\}$ is an alphabet of size $(n-1)$. The two classes are:

I. Strings that **do not contain** n. These are all (non repeating) strings of length k from $A - \{n\}$, of which there are $P(n-1,k)$.

II. Strings that **do contain** n. We can form these strings by making the following choices in succession.

(1) Choose a (non repeating) string of length $(k-1)$ from $A - \{n\}$: #ways $= P(n-1,k-1)$.

(2) This string has k spaces between it's symbols. (e.g. "_a_b_c_d_" for $k = 5$). Choose one of those spaces in which to insert the symbol n. #ways $= k$.

By the multiplication principle, the number of such strings containing n is $k \cdot P(n-1,k-1)$.

By the addition principle, the number of strings in both classes is $k \cdot P(n-1,k-1) + P(n-1,k)$ which is the right hand side of the above formula. ■