CSE 16
Homework Assignment 7
Solutions to Selected Problems

<u>Chapter 10: #42</u>
Prove that the n^{th} Fibonacci number F_n is even if and only if $3|n$.

Proof:
We use strong induction form IId with two base cases to show: $\forall\, n \geq 1\colon 2|F_n \Leftrightarrow 3|n$

For $n = 1$ we have $F_1 = 1$ is odd and $3 \nmid 1$. For $n = 2$ we have $F_2 = 1$ is odd and $3 \nmid 2$. In both cases $2|F_n \Leftrightarrow 3|n$ is a true statement since both sides are false. The base cases are now complete.

Let $n > 2$ be chosen arbitrarily. Assume for all k in the range $1 \leq k < n$ that $2|F_k \Leftrightarrow 3|k$. We must show that $2|F_n \Leftrightarrow 3|n$. The Fibonacci recurrence says that

$$F_n = F_{n-1} + F_{n-2}.$$

Observe that given any three consecutive integers $\{n-2, n-1, n\}$, exactly one of them is divisible by 3. We have two cases to consider.

<u>Case 1</u>: $3|n$
In this case, the above observation impies that $3 \nmid (n-1)$ and $3 \nmid (n-2)$. By the induction hypothesis (with $k = n-1$ and $k = n-2$), we have both F_{n-1} and F_{n-2} are odd. But $\text{odd} + \text{odd} = \text{even}$, so that $F_n = F_{n-1} + F_{n-2}$ is even. The statement $2|F_n \Leftrightarrow 3|n$ is therefore true, since both sides are true.

<u>Case 2</u>: $3 \nmid n$
In this case, our observation implies that either $3|(n-1)$ and $3 \nmid (n-2)$, or $3 \nmid (n-1)$ and $3|(n-2)$. By the induction hypothesis (with $k = n-1$ and $k = n-2$) we obtain that either F_{n-1} is even and F_{n-2} is odd or F_{n-1} is odd and F_{n-2} is even. Notice though that $\text{even} + \text{odd} = \text{odd} + \text{even} = \text{odd}$, so that $F_n = F_{n-1} + F_{n-2}$ must be odd. The statement $2|F_n \Leftrightarrow 3|n$ is again true, since both sides are false.

In both cases $2|F_n \Leftrightarrow 3|n$ is true, as required. The result follows for all $n \geq 1$ by the 2nd PMI. ■

<u>(11.2) #2</u>
Consider the relation $R = \{(a,b),(a,c),(c,c),(b,b),(c,b),(b,c)\}$ on the set $A = \{a,b,c\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

Solution:

<u>Reflexive</u>: **No**
R does not contain (a,a), so that $a \not R a$.

<u>Symmetric</u>: **No**
R contains (a,b) but not (b,a), so $aRb \rightarrow bRa$ is false.

<u>Transitive</u>: **Yes**

■

(11.3) #12
Prove or disprove: If R and S are two equivalence relations on a set A, then $R \cup S$ is also an equivalence relation on A.

Solution:
The statement is **false**, which we now prove by a counterexample.

Let $A = \{1, 2, 3\}$, $R = \{(1,1), (2,2), (1,2), (2,1)\}$ and $S = \{(1,1), (3,3), (1,3), (3,1)\}$. One checks that both R and S are equivalence relations. However, while the relation

$$R \cup S = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\},$$

is both reflexive and symmetric, it is **not** transitive. To see this observe $2(R \cup S)1$ and $1(R \cup S)3$, but $2\cancel{(R \cup S)}3$. ■

Additional Problems #2
Let $A = \{ (a,b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \}$. Define a relation R on A by

$$(a,b)R(c,d) \text{ iff } ad = bc.$$

In other words

$$R = \left\{ \big((a,b),(c,d)\big) \mid a,b,c,d \in \mathbb{Z} \text{ and } b \neq 0, d \neq 0, ad = bc \right\}$$

Prove that R is an equivalence relation.

Proof:
Reflexive:
Given $(a,b) \in A$, we have $ab = ba$, hence $(a,b)R(a,b)$ and R is reflexive.

Symmetric:
Given $(a,b),(c,d) \in A$, we have: $(a,b)R(c,d) \to ad = bc \to cb = da \to (c,d)R(a,b)$, and hence R is symmetric.

Transitive:
Let (a,b), (c,d), $(e,f) \in A$, and suppose $(a,b)R(c,d)$ and $(c,d)R(e,f)$. Then $ad = bc$ and $cf = de$. Multiply the first equation by f and the second equation by b to get $adf = bcf$ and $bcf = bde$. It follows that $adf = bde$. Cancel d from both sides (which is $\neq 0$ by the definition of A) to get $af = be$. Therefore $(a,b)R(e,f)$ as required. ■

Remarks:
Notice that the requirement that $b \neq 0$ in the definition of A, though not needed for the reflexive and symmetric properties, is necessary for the transitive property. Let us denote the equivalence class of $(a,b) \in A$ by $[a,b]$, and denote the set of equivalence classes of R by Q. We can then define four arithmetic operations on Q as follows.

$$[a,b] + [c,d] = [ad + bc, bd]$$
$$[a,b] - [c,d] = [ad - bc, bd]$$
$$[a,b] * [c,d] = [ac, bd]$$
$$\frac{[a,b]}{[c,d]} = [ad, bc]$$

where the last operation is valid only if $c \neq 0$. The equivalence relation R discussed in this problem, along with the above arithmetic operations, constitute the standard method for constructing the rational numbers $\mathbb{Q}$ from the integers $\mathbb{Z}$. A rational number $\frac{a}{b}$ is thereby defined to be an equivalence class $[a, b]$ of R.

Exercise:
Prove that the above operations are well-defined. In other words, the definitions do not depend on the particular representatives of the equivalence classes used in the formulas. For instance, in the addition operation, if $(a, b)R(a', b')$ and $(c, d)R(c', d')$ (so that $[a, b] = [a', b']$ and $[c, d] = [c', d']$), then $[ad + bc, bd] = [a'd' + b'c', b'd']$. Likewise for the other three operations.