CSE 16
Homework Assignment 5
Solutions to Selected Problems

Chapter 4 (p. 126): #26

Every odd integer is a difference of two squares. (Example $7^2 = 4^2 - 3^2$ etc.)

Proof:

Let n be an odd integer, so that $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then

$$(k+1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = n$$

Showing that n is the difference of two squares, as claimed. ■

Chapter 5 (p. 136): #8

Suppose $x \in \mathbb{R}$. If $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$, then $x \geq 0$.

Proof:

Let $x \in \mathbb{R}$. We show that $x < 0 \implies x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$, and the result follows by contraposition.

Assume x is negative. Then any odd power of x is negative, and any even power of x is positive. It follows that the terms x^5, $3x^3$ and $3x$ are all negative, and the terms $-4x^4$, $-x^2$ and -4 are negative. Therefore, the sum of these terms is also negative, i.e. $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$. We have shown

$$x < 0 \implies x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0,$$

and hence

$$x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0 \implies x \geq 0.$$

■

Chapter 6 (p. 144): #8
Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Proof:
Assume $a^2 + b^2 = c^2$. Assume (to get a contradiction) that both a and b are odd. Then $a = 2k_1 + 1$ and $b = 2k_2 + 1$, for some $k_1, k_2 \in \mathbb{Z}$. We have

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (4k_1^2 + 4k_1 + 1) + (4k_2^2 + 4k_2 + 1) \\ &= 2(2k_1^2 + 2k_1 + 2k_2^2 + 2k_2) + 2 \\ &= 2[2(k_1^2 + k_1 + k_2^2 + k_2) + 1] \end{aligned}$$

Thus c^2 is even. It follows that c is even (by a result proved in class), hence $c = 2m$ for some $m \in \mathbb{Z}$, and hence $c^2 = 4m^2$. Therefore

$$4m^2 = 2[2(k_1^2 + k_1 + k_2^2 + k_2) + 1].$$

Dividing the above equation by 2 gives

$$2m^2 = 2(k_1^2 + k_1 + k_2^2 + k_2) + 1$$

in which the LHS is even while the RHS is odd. This contradiction shows that our assumption was false, and therefore either a is even or b is even. ■

Chapter 6 (p. 144): #24
The number $\log_2 3$ is irrational.

Proof:
Assume (to get a contradiction) that $\log_2 3$ is rational, i.e. there exist $a, b \in \mathbb{Z}$ such that

$$\begin{aligned} & \log_2 3 = \frac{a}{b} \\ \therefore \quad & 2^{\log_2 3} = 2^{a/b} \\ \therefore \quad & 3 = 2^{a/b} \\ \therefore \quad & 3^b = \left(2^{a/b}\right)^b = 2^a. \end{aligned}$$

In this last equation the LHS is odd while the RHS is even, a contradiction. (Note: another contradiction is that the last equation violates the theorem on the uniqueness of prime factorization.) We conclude that our assumption was false, and therefore $\log_2 3$ is irrational. ■