**CSE 16**
**Homework Assignment 4**
**Solutions to Selected Problems**

**<u>3.7 #4a</u>**
How many 4-letter lists (with repetition allowed) from the set $\{T, H, E, O, R, Y\}$ are there that *don't* begin with T, or *don't* end in Y?

**Solution:**
Let
$$A = \{\text{lists of length 4 (with repetition) not beginning with } T\}$$
and
$$B = \{\text{lists of length 4 (with repetition) not ending with } Y\}.$$

We seek to find $|A \cup B|$. Since the alphabet in this case contains 6 letters, the multiplication principle gives $|A| = 5 \cdot 6 \cdot 6 \cdot 6 = 1080$, $|B| = 6 \cdot 6 \cdot 6 \cdot 5 = 1080$ and $|A \cap B| = 5 \cdot 6 \cdot 6 \cdot 5 = 900$. By the inclusion-exclusion principle, we have $|A \cup B| = 1080 + 1080 - 900 = \mathbf{1,260}$. ■

**Alternate Solution:**
Let $S = \{\text{all lists of length 4 (with repetition) from the given 6-element set}\}$, so $|S| = 6^4 = 1296$. Let $A$ and $B$ be as above. Observe that by DeMorgan's law $\overline{\bar{A} \cap \bar{B}} = \bar{\bar{A}} \cup \bar{\bar{B}} = A \cup B$, where all complements are taken relative to $S$. Also note that $\bar{A} \cap \bar{B}$ consists of those lists both beginning with $T$ and ending with $Y$. The multiplication principle gives $|\bar{A} \cap \bar{B}| = 6 \cdot 6 = 36$, and therefore

$$|A \cup B| = |\overline{\bar{A} \cap \bar{B}}| = |S| - |\bar{A} \cap \bar{B}| = 1296 - 36 = \mathbf{1,260}.$$

■

**3.8 #6**

A bag contains 20 identical red balls, 20 identical blue balls, 20 identical green balls, one white ball, and one black ball. You reach in and grab 20 balls. How many different outcomes are possible?

**Solution:**

Observe that there might as well be an unlimited supply of Red, Blue and Green balls, while the White and Black balls are special, since their supply is limited to one each. We partition the set of outcomes into 4 disjoint types according to the numbers of White and Black balls present, as follows.

(1) No White and no Black. In this case the outcome is a 20-element multiset based on the 3-element set $\{R, B, G\}$. There are $\binom{20+3-1}{20} = \binom{22}{20} = 231$ of these.

(2) One White and no Black. The remaining balls form a 19-element multiset from a 3-element set, of which there are $\binom{19+3-1}{19} = \binom{21}{19} = 210$.

(3) No White and one Black. Again the remaining balls form a 19-element multiset from a 3-element set, and again there are $210$ outcomes of this type.

(4) One White and one Black. The remaining balls form an 18-element multiset from a 3-element set, of which there are $\binom{18+3-1}{18} = \binom{20}{18} = 190$.

By the addition principle, the total number of outcomes is $231 + 210 + 210 + 190 = \mathbf{841}$. ■

**3.9 #6**

Given a sphere S, a *great circle* of S is the intersection of S with a plane through its center. Every great circle divides S into two parts. A *hemisphere* is the union of the great circle and one of these two parts. Show that if five points are placed arbitrarily on S, then there is a hemisphere that contains four of them.

[Hint: two points on a sphere determine a great circle, namely the intersection of the sphere with the plane containing the two points and the center of the sphere.]

**Solution:**

Call the five points $A$, $B$, $C$, $D$ and $E$. According to the hint, the points $A$ and $B$ determine a great circle on the sphere, which we may call the $equator$. The equator itself determines two hemispheres, which we may call the *northern* and *southern* hemispheres, respectively. Note that points $A$ and $B$ lie in both hemispheres. The remaining points $C$, $D$ and $E$ each lie in either the northern or southern hemispheres (or both if they lie on the equator.) By the pigeonhole principle, when 3 objects (points $C$, $D$, $E$) are placed in 2 boxes (hemispheres), at least one box (hemisphere) contains at least two objects (points). By relabeling points and hemispheres if necessary, we may conclude that points $C$ and $D$ lie in the northern hemisphere. But then we have 4 points, $A$, $B$, $C$ and $D$, all lying in one hemisphere, as required. ■

**3.10 #10**

Use a combinatorial proof to show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

**Proof:**

Let $S$ be a set with $|S| = n$. We show that both sides of the above identity count the cardinality of the following finite set.

$$\{\, (x, A) \mid x \in A \subseteq S \,\}$$

Equivalently, we must determine the number of ways to select, from a set of $n$ people, a committee of any size, together with a special committee member designated as chairperson. (Note the key difference between this problem and 3.6 #10, whose combinatorial proof can be found on page 5 of the class notes from 7-13-22, is that here the size of the committee is not fixed.) We count the above set of pairs in two ways.

The first method is to (1) select any $x \in S$ (which can be done in $n$ ways), then (2) select a subset of $S - \{x\}$ of any size (which can be done in $2^{n-1}$ ways.) By the multiplication principle, we can perform this sequence of choices in $n2^{n-1}$ ways, which is therefore the number of pairs in the above set.

The second method is to partition $S$ based on the size of the subset $A$ (i.e. the committee), and use the addition principle. For each $k$ in the range $1 \le k \le n$, we (1) pick a subset $A \subseteq S$ with $|A| = k$, then (2) pick an element $x \in A$. Step (1) can be performed in $\binom{n}{k}$ ways, and (2) can be done in $k$ ways. By the multiplication principle, this sequence can be performed in $k\binom{n}{k}$ ways. Summing over all such $k$, we have that the total number of pairs in the above set is $\sum_{k=1}^{n} k\binom{n}{k}$, establishing the identity

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

■

[**Note to graders**: an algebraic proof (like the one below) should be awarded at most half credit, since the problem statement calls for a combinatorial proof.]

**Algebraic Proof:**

This proof uses the binomial theorem, and a little calculus. For any $x \in \mathbb{R}$ and $n \ge 0$, we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Upon differentiating both sides of this equation with respect to $x$, we obtain

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}.$$

Now setting $x = 1$ yields the identity $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$, as required. ■