

CSE 16

Homework Assignment 3

Solutions to Selected Problems

(3.3) 4

Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such lineups are there in which exactly one of the 5 cards is a queen?

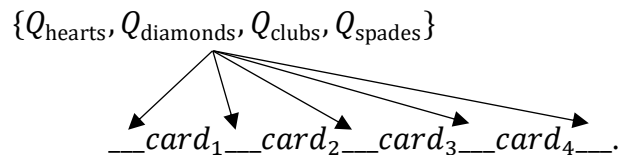
Solution:

We seek the number of 5-permutations of a 52-card deck containing exactly one queen. We can construct such a permutation by performing the following two steps in succession.

- (1) Choose a 4-permutation containing no queens, i.e. a 4 permutation from a 48-element set (the non-queens). The number of ways this can be done is

$$\frac{48!}{(48-4)!} = \frac{48!}{44!} = 48 \cdot 47 \cdot 46 \cdot 45 = 4,669,920$$

- (2) Choose one of the 5 spaces between the existing 4 cards, and place one of the 4 queens in that space. The number of ways this can be done is $5 \cdot 4 = 20$.



By the multiplication principle, the total number of 5-card lineups containing exactly one queen is $(4669920) \cdot (20) = \mathbf{93,398,400}$.

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Alternate Solution:

We can also do the following three steps.

- (1) Choose a queen: #ways = 4
- (2) Choose a 4-element subset of the non-queens: #ways = $\binom{48}{4} = 194,580$
- (3) Choose a permutation of our 5 chosen cards: #ways = $5! = 120$

The multiplication principle gives the number of required lineups as: $4 \cdot (194580) \cdot 120 = \mathbf{93,398,400}$.

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(3.4) 10

How many permutations of the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are there in which the digits alternate even and odd? (For example, 2183470965.)

Solution:

To construct such a permutation, we choose a permutation of the even digits $\{0, 2, 4, 6, 8\}$ (#ways = $5!$), then choose a permutation of the odd digits $\{1, 3, 5, 7, 9\}$ (#ways = $5!$), then interleave the two permutations. There are 2 ways to interleave. Even first: even, odd, even, odd, ..., and odd first: odd, even, odd, even, By the multiplication principle, the total number of alternating digit strings is

$$5! \cdot 5! \cdot 2 = (120)^2 \cdot 2 = (14,400) \cdot 2 = \mathbf{28,800}.$$

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(3.5) 16

How many 6-element subsets of $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ have exactly three even elements? How many do not have exactly three even elements?

Solution:

We can form a 6-element subset of A with exactly 3 even elements by choosing a 3-element subset of the even elements $\{0, 2, 4, 6, 8\}$, and combining it with a 3-element subset subset of the odds $\{1, 3, 5, 7, 9\}$. By the multiplication principle,

$$(\# \text{ of 6-subsets of } A \text{ with exactly 3 even elements}) = \binom{5}{3} \cdot \binom{5}{3} = 10^2 = \mathbf{100}.$$

The total number of 6-element subsets of A is $\binom{10}{6} = 210$, and therefore by the subtraction principle

$$(\# \text{ of 6-subsets of } A \text{ not having exactly 3 evens}) = 210 - 100 = \mathbf{110}.$$

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(3.6) 4

Use the binomial theorem to find the coefficient of x^6y^3 in $(3x - 2y)^9$.

Solution:

The binomial theorem says that $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$. Letting $a = 3x$, $b = -2y$ and $n = 9$ gives

$$\begin{aligned} (3x - 2y)^9 &= \sum_{k=0}^9 \binom{9}{k} \cdot 3^{9-k} \cdot x^{9-k} \cdot (-2)^k \cdot y^k \\ &= \sum_{k=0}^9 (-1)^k 2^k 3^{9-k} \binom{9}{k} x^{9-k} y^k \end{aligned}$$

The monomial x^6y^3 occurs when $k = 3$, and therefore the coefficient of x^6y^3 is

$$(-1)^3 2^3 3^6 \binom{9}{3} = -8 \cdot 729 \cdot 84 = \mathbf{-489,888}.$$

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