CSE 16
Homework Assignment 3
Solutions to Selected Problems

<u>**(3.3) 4**</u>
Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such lineups are there in which exactly one of the 5 cards is a queen?

Solution:
We seek the number of 5-permutations of a 52-card deck containing exactly one queen. We can construct such a permutation by performing the following two steps in succession.

(1) Choose a 4-permutation containing no queens, i.e. a 4 permutation from a 48-element set (the non-queens). The number of ways this can be done is

$$\frac{48!}{(48-4)!} = \frac{48!}{44!} = 48 \cdot 47 \cdot 46 \cdot 45 = 4{,}669{,}920$$

(2) Choose one of the 5 spaces between the existing 4 cards, and place one of the 4 queens in that space. The number of ways this can be done is $5 \cdot 4 = 20$.

$$\{Q_{\text{hearts}}, Q_{\text{diamonds}}, Q_{\text{clubs}}, Q_{\text{spades}}\}$$

$$\_\_ card_1 \_\_ card_2 \_\_ card_3 \_\_ card_4 \_\_.$$

By the multiplication principle, the total number of 5-card lineups containing exactly one queen is $(4669920) \cdot (20) = \mathbf{93{,}398{,}400.}$

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Alternate Solution:
We can also do the following three steps.

(1) Choose a queen: $\#\text{ways} = 4$
(2) Choose a 4-element subset of the non-queens: $\#\text{ways} = \binom{48}{4} = 194{,}580$
(3) Choose a permutation of our 5 chosen cards: $\#\text{ways} = 5! = 120$

The multiplication principle gives the number of required lineups as: $4 \cdot (194580) \cdot 120 = \mathbf{93{,}398{,}400}$.

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(3.4) 10

How many permutations of the digits $\{0,1,2,3,4,5,6,7,8,9\}$ are there in which the digits alternate even and odd? (For example, 2183470965.)

Solution:

To construct such a permutation, we choose a permutation of the even digits $\{0,2,4,6,8\}$ (#ways $= 5!$), then choose a permutation of the odd digits $\{1,3,5,7,9\}$ (#ways $= 5!$), then interleave the two permutations. There are 2 ways to interleave. Even first: even, odd, even, odd, …, and odd first: odd, even, odd, even, … . By the multiplication principle, the total number of alternating digit strings is

$$5! \cdot 5! \cdot 2 = (120)^2 \cdot 2 = (14{,}400) \cdot 2 = \mathbf{28{,}800.}$$

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(3.5) 16

How many 6-element subsets of $A = \{0,1,2,3,4,5,6,7,8,9\}$ have exactly three even elements? How many do not have exactly three even elements?

Solution:

We can form a 6-element subset of A with exactly 3 even elements by choosing a 3-element subset of the even elements $\{0,2,4,6,8\}$, and combining it with a 3-element subset subset of the odds $\{1,3,5,7,9\}$. By the multiplication principle,

$$(\text{\# of 6-subsets of } A \text{ with exactly 3 even elements}) = \binom{5}{3} \cdot \binom{5}{3} = 10^2 = \mathbf{100.}$$

The total number of 6-element subsets of A is $\binom{10}{6} = 210$, and therefore by the subtraction principle

$$(\text{\# of 6-subsets of } A \text{ not having exactly 3 evens}) = 210 - 100 = \mathbf{110.}$$

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(3.6) 4

Use the binomial theorem to find the coefficient of x^6y^3 in $(3x - 2y)^9$.

Solution:

The binomial theorem says that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Letting $a = 3x$, $b = -2y$ and $n = 9$ gives

$$(3x - 2y)^9 = \sum_{k=0}^{9} \binom{9}{k} \cdot 3^{9-k} \cdot x^{9-k} \cdot (-2)^k \cdot y^k$$

$$= \sum_{k=0}^{9} (-1)^k 2^k 3^{9-k} \binom{9}{k} x^{9-k} y^k$$

The monomial x^6y^3 occurs when $k = 3$, and therefore the coefficient of x^6y^3 is

$$(-1)^3 2^3 3^6 \binom{9}{3} = -8 \cdot 729 \cdot 84 = \mathbf{-489{,}888}.$$

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