Problem (3.10) #9

Let $0 \le m \le n$. Then

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

First let's see why this should be true.

$$\binom{n+1}{m+1} = \binom{n}{m} + \binom{n}{m+1} \quad \text{(by Pascal's Iden.)}$$

$$= \binom{n}{m} + \binom{n-1}{m} + \binom{n-1}{m+1} \quad \text{(by Pascal on the last term)}$$

$$= \binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \binom{n-2}{m+1} \quad \text{"}$$

$$\vdots$$

$$= \binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m+1}{m} + \binom{m+1}{m+1}$$

this is $= 1$, so also eq to $\binom{m}{m}$

$$= \binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m+1}{m} + \binom{m}{m}$$

This can be made into a proper proof by a technique which we will study later, called induction.

Before we do a combinatorial proof, let's look at what it says about Pascal's triangle.

$m = 0$, $2$, $3$, $4$, $5$, $6$, $7$

| $n=$ | |
|---|---|
| 0 | 1 |
| 1 | 1 1 |
| 2 | 1 2 1 |
| 3 | 1 3 3 1 |
| 4 | 1 4 6 4 1 |
| 5 | 1 5 10 10 5 1 |
| 6 | 1 6 15 $\boxed{20}$ 15 6 1 |
| 7 | 1 7 21 $\boxed{35}$ 35 21 7 1 |
| 8 | 1 8 28 56 70 $\boxed{56}$ 28 8 1 |

special case: $n=5$, $m=2$. Let $S=\{1,2,3,4,5,6\}$. Then $\binom{6}{3}=\binom{n+1}{m+1}$ is the # of 3-subsets of $S$. We can count these another way:

| $k$ | | # choices |
|---|---|---|
| 2 | • $\{1,2,3\}=\{1,2\}\cup\{3\}$ | $\binom{2}{2}=1$ |
| 3 | • (any 2-subset of $\{1,2,3\}$) $\cup\{4\}$ | $\binom{3}{2}=3$ |
| 4 | • (any 2-subset of $\{1,2,3,4\}$) $\cup\{5\}$ | $\binom{4}{2}=6$ |
| 5 | • (any 2-subset of $\{1,2,3,4,5\}$) $\cup\{6\}$ | $\binom{5}{2}=10$ |

$\therefore\quad 1 + 3 + 6 + 10 = 20$

i.e. $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}=\binom{6}{3}$

To generalize the above special case we introduce some notation

$$[n] = \{1, 2, 3, \ldots, n\}$$

Proof of theorem

Let $S = [n+1]$, so $|S| = n+1$. There are $\binom{n+1}{m+1}$ subsets of $S$ of size $(m+1)$.

These subsets of $S$ partition into $n-m+1$ (pairwise disjoint) types, as follows.

| type | | # choices |
|---|---|---|
| $m$ | $[m] \cup \{m+1\}$ | $\binom{m}{m} = 1$ |
| $m+1$ | (any $m$-subset of $[m+1]$) $\cup \{m+2\}$ | $\binom{m+1}{m}$ |
| $m+2$ | (any $m$-subset of $[m+2]$) $\cup \{m+3\}$ | $\binom{m+2}{m}$ |
| $m+3$ | (any $m$-subset of $[m+3]$) $\cup \{m+4\}$ | $\binom{m+3}{m}$ |
| $\vdots$ | | $\vdots$ |
| $n-1$ | (any $m$-subset of $[n-1]$) $\cup \{n\}$ | $\binom{n-1}{m}$ |
| $n$ | (any $m$-subset of $[n]$) $\cup \{n+1\}$ | $\binom{n}{m}$ |

By the addition principle, the # of $(m+1)$-subsets of $S = [n+1]$ is

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}. \quad \blacksquare$$