CSE 16 8-4-22

- SETs: if response $\geq 85\%$, then $+ 0.1\%$
- quiz 6 today 4-10 pm
- hw6: reviews end Sat. 10 pm
- hw7: additional Problems

Also: $\displaystyle\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$ $(0 \leq m \leq n)$

induction on n

Base case: $n = m$.

(11.3) Equivalence Relations

Defn A relation $R \subseteq A \times A$ on A is called an equivalence relation iff it is

(1) reflexive
(2) symmetric
(3) transitive

Recall: equiv. relations on $\mathbb{Z}$!

- $=$ i.e. $\{(x,x) \mid x \in \mathbb{Z}\}$
- $\equiv_m$ i.e. $\{(x,y) \mid m \mid (x-y)\}$

Complete exercise for $\equiv_m$

- reflexive: ✓

$$m \mid 0 \rightarrow m \mid (x-x) \rightarrow x \equiv_m x$$

- symmetric: ✓

note: $m \mid a \rightarrow m \mid (-a)$ since $a = km \Rightarrow -a = (-k)m$

Thus

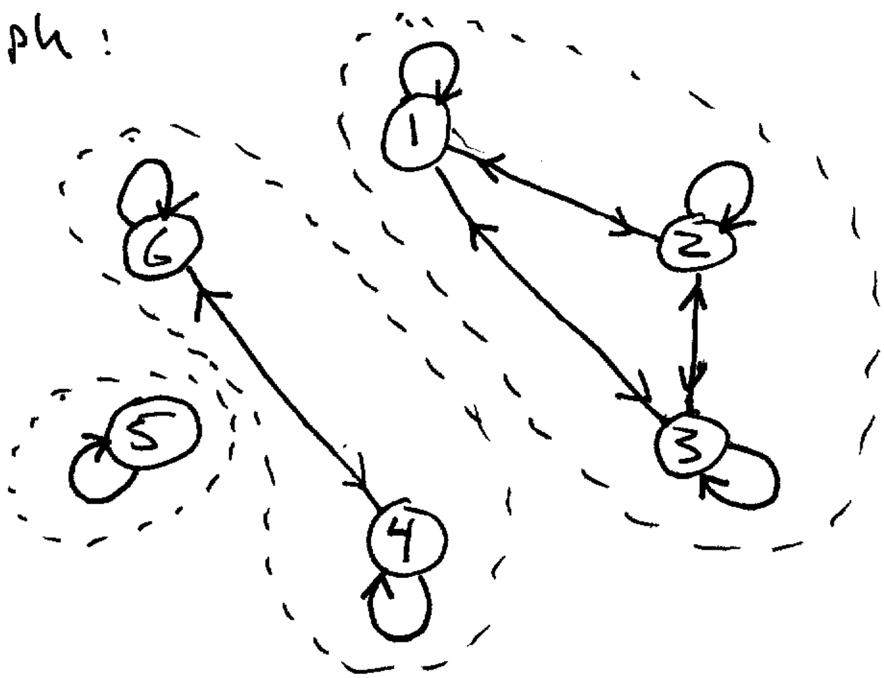
$$\begin{aligned} x \equiv_m y &\rightarrow m \mid (x-y) \\ &\rightarrow m \mid (y-x) \\ &\rightarrow y \equiv_m x \end{aligned}$$

- transitive: done last time ✓

Ex. let $A = \{1, 2, 3, 4, 5, 6\}$ and

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,6), (5,5), (6,4), (6,6)\}$

digraph:



check: ref, symm, trans.

Defn

let $R \subseteq A \times A$ be an equiv. rel., and let $a \in A$. The equivalence class containing a, denoted $[a]$, is the set

$$[a] = \{x \in A \mid xRa\} \subseteq A$$

i.e. $[a]$ is the set of all $x \in A$ that are related to a (under R)

other notation: $[a]_R$

Ex. (Previous) $A = \{1, 2, 3, 4, 5, 6\}$

$[1] = [2] = [3] = \{1, 2, 3\}$

$[4] = [6] = \{4, 6\}$

$[5] = \{5\}$

Lemma

Let R be an eq. rel. on A. Then

$$aRb \underset{\Leftarrow}{\overset{\Rightarrow}{\text{iff}}} [a]=[b]$$

for any $a, b \in A$.

Proof

($\Rightarrow$) Suppose aRb. It's suff. to show $x \in [a] \Leftrightarrow x \in [b]$, for then $[a]=[b]$

$x \in [a] \Rightarrow xRa$ (def.)

$\Rightarrow xRa \wedge aRb$ (hyp.)

$\Rightarrow xRb$ (trans.) $\Leftarrow$

$\Rightarrow x \in [b]$ (def.)

also

$x \in [b] \Rightarrow xRb$ (def.)

$\Rightarrow xRb \wedge aRb$ (hyp.)

$\Rightarrow xRb \wedge bRa$ (symm.) $\Leftarrow$

$$\Rightarrow xRa \qquad \text{(trans.)}$$

$$\Rightarrow x \in [a] \qquad \text{(def.)}$$

$\therefore\ [a] = [b]$

($\Leftarrow$) Suppose $[a] = [b]$. Then aRa since R is reflexive. Hence

$$a \in [a] \qquad \text{(def.)}$$

$$\Rightarrow a \in [b] \qquad \text{(hyp.)}$$

$$\Rightarrow aRb \qquad \text{(def.)}$$

$\blacksquare$

<u>notice</u>: all 3 properties of R were used.

Given $m \in \mathbb{N}$, what are eq. classes of $\equiv_m$?

By the lemma

$$
\begin{aligned}
x \in [a]_m \text{ iff } & x \equiv_m a \\
\text{iff } & m \mid (x-a) \\
\text{iff } & x-a = km \quad (\text{some } k \in \mathbb{Z}) \\
\text{iff } & x = km + a
\end{aligned}
$$

so $[a]_m = \{ x \in \mathbb{Z} \mid x = km + a \text{ for some } k \in \mathbb{Z} \}$

<u>Ex.</u> $m = 2$

$x \in [0]$ iff $x = 2k$ iff x even

$x \in [1]$ iff $x = 2k+1$ iff x odd

Thus

$[0] = [\pm 2] = [\pm 4] = \cdots = \{\text{even inte.}\}$

$[\pm 1] = [\pm 3] = [\pm 5] = \cdots = \{\text{odd inte.}\}$.

Ex. $m = 3$

$x \in [0]$ iff $x = 3k$

$x \in [1]$ iff $x = 3k+1$

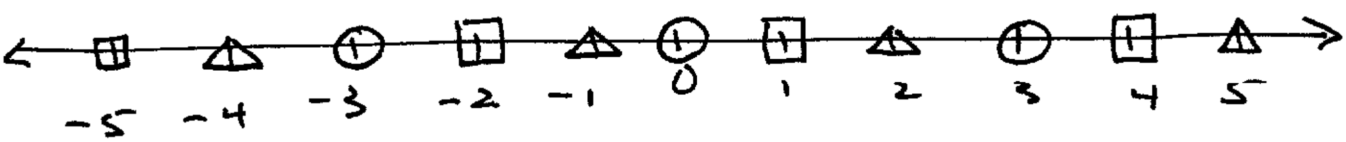
$x \in [2]$ iff $x = 3k+2$

Thus

$[0] = [3] = [6] = \cdots = \{3k \mid k \in \mathbb{Z}\}$ $\quad \bigcirc$

$[1] = [4] = [7] = \cdots = \{3k+1 \mid k \in \mathbb{Z}\}$ $\quad \square$

$[2] = [5] = [8] = \cdots = \{3k+2 \mid k \in \mathbb{Z}\}$ $\quad \triangle$



In general, for any $m \in \mathbb{N}$, we can write $x \in \mathbb{Z}$ as

$$x = km + r \quad (0 \leq r < m)$$

by div. algorithm. Then $x \in [r]$

Thus for any r : $0 \le r < m$

$$[r] = [m+r] = [2m+r] = \cdots = \{km+r \mid k \in \mathbb{Z}\}$$

so : $[0], [1], [2], \ldots, [m-1]$ are

all of the eq. classes of $\equiv_m$.

(11.4) Equivalence Classes and Partitions

Recall a Partition of a set A is a set of subsets of A : $P \subseteq \mathbb{P}(A)$, say $P = \{S_1, S_2, S_3, \ldots\}$ where

$$S_i \subseteq A$$

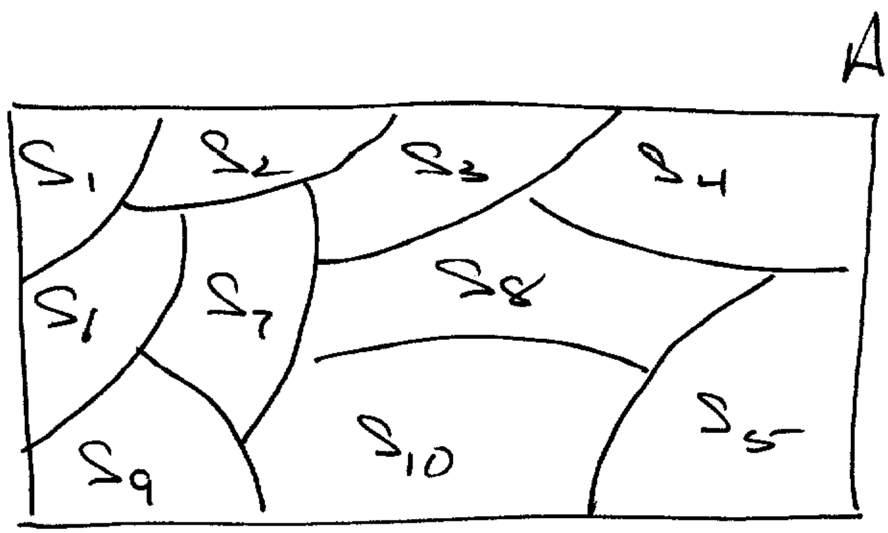
satisfies

(1) $S_i \neq \emptyset$

(2) if $i \neq j$, then $S_i \cap S_j = \emptyset$

(3) $\bigcup_i S_i = A$

Picture:



Theorem

Let R be an eq. relation on A. Then the set of eq. classes of R

$$\{[a] \mid a \in A\}$$

forms a Partition of A.

Proof

for any $a \in A$ we have $a \in [a]$ since aRa by reflexivity. This shows $[a] \neq \phi$, which proves (1).

similarly $\{a\} \subseteq [a]$, so

$$A = \bigcup_{a\in A}\{a\} \subseteq \bigcup_{a\in A}[a] \subseteq A$$

$\therefore$ $A = \bigcup_{a\in A} [a]$, proving (3).

Finally we show $[a]\neq[b] \Rightarrow [a]\cap[b]=\phi$ for any $a,b\in A$, establishing (2).

We prove this by contraposition, i.e. we show

$$[a]\cap[b]\neq\phi \implies [a]=[b]$$

Assume $c\in[a]\cap[b]$. Then

$c\in[a] \Rightarrow cRa \Rightarrow aRc$ (symm.)
$c\in[b] \Rightarrow cRb$

$\therefore$ aRb (trans.)

$\therefore$ $[a]=[b]$ by lemma in preceding section.

This proves (2). $\blacksquare$

<u>Theorem</u>

Given a Partition

$$P=\{S_i \subseteq A \mid i=1,2,3,\dots\}$$

of A, there exists a unique equivalence class R on A whose eq. classes $[a]$ are exactly the members S_i of P, i.e.

$$P=\{[a]_R \mid a \in A\}$$

Proof:

Define $R \subseteq A \times A$ by

$$R = \{ (a,b) \mid \exists i : a,b \in S_i \}.$$

In other words aRb iff both a and b belong to the same member of Partition P.