cse 16 8-3-22

- SETs (student eval. of teaching)

  <u>open</u>: Thur 8/4 7:00am

  <u>close</u>: Tue 8/9 11:59 PM

<u>Ex.</u> P.6 last time notes

$$\forall n \geq 1: \boxed{\sum_{k=1}^{n} F_{2k-1} = F_{2n}}$$

<u>Proof.</u>

I. done

IIa. $\forall n \geq 1: P(n) \rightarrow P(n+1)$.

let $n \geq 1$. Assume

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n}$$

must show

$$\sum_{k=1}^{n+1} F_{2k-1} = F_{2n+2} .$$

so

$$\sum_{k=1}^{n+1} F_{2k-1} = \left(\sum_{k=1}^{n} F_{2k-1}\right) + F_{2(n+1)-1}$$

$$= F_{2n} + F_{2n+1} \quad \Big\{ \text{by ind. hyp.}$$

$$= F_{2n+2} . \qquad \blacksquare$$

<u>Exercises</u>

- <u>ch 10 #25</u> : $\forall n \geq 1 : \sum_{k=1}^{n} F_k = F_{n+2} - 1$ (weak)
- <u>ch 10 #29</u> : $\forall n \geq 0 : \sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$ (strong)

[Hint: use $\binom{a}{b} = 0$ when $b < 0$ or $b > a$.]

Chapter 11 : Relations

(11.1) General relations

Defn

A relation from a set A to a set B is a subset : $R \subseteq A \times B$

notation.

given $x \in A$, $y \in B$ write

$$x R y \text{ to mean } (x, y) \in R$$

and

$$x \not{R} y \text{ to mean } (x, y) \notin R$$

we call A the Domain and B the codomain of R.

more generally, a relation on sets $A_1, A_2, \ldots, A_n$ is

$$R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_n$$

(used in theory of data bases.)

Defn

A relation on A is a relation from A to A : $R \subseteq A \times A$.

Examples of relations on $\mathbb{Z}$:

- $<$ is $\{(x,y) \mid x < y\} \subseteq \mathbb{Z} \times \mathbb{Z}$

  so $(1,2) \in <$ and $(2,1) \notin <$

- $\leq$ .... $x \leq y$ ....
- $>$ - - - - - - - - - $x > y$ - - - - - -
- $\geq$ - - - - - - - - - $x \geq y$ - - - - - - -

- $=$ is $\{(x,x) \mid x \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$

- $\neq$ is $\mathbb{Z} \times \mathbb{Z} - \{(x,x) \mid x \in \mathbb{Z}\}$

- $\mid$ is $\{(x,y) \mid x \mid y\} \subseteq \mathbb{Z}^* \times \mathbb{Z}$
  where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$.

- let $m \in \mathbb{N}$. Then
  $\equiv_m$ is $\{(x,y) \mid m \mid (x-y)\} \subseteq \mathbb{Z} \times \mathbb{Z}$

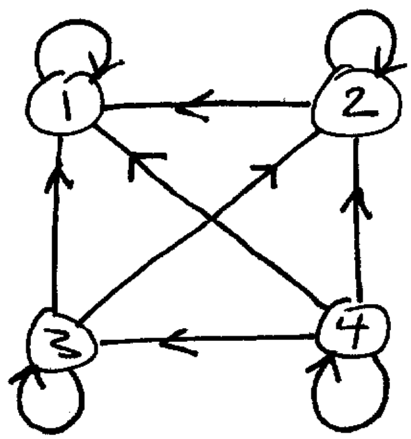
Relations on finite sets

Ex $A = \{1, 2, 3, 4\}$ and

$$R = \{(1,1), (2,1), (2,2), (3,3), (3,2), (3,1), (4,4), (4,3), (4,2), (4,1)\}$$

Thus: $4R1$, $3R2$ but $3 \not R 4$.

Directed graph representation:



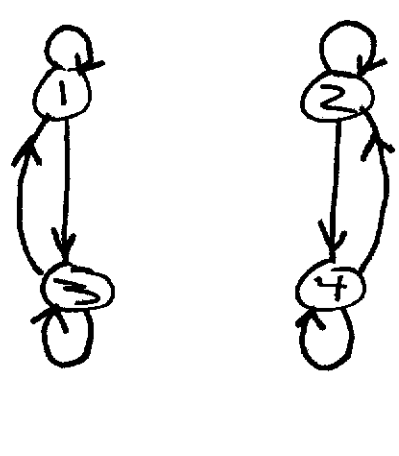
This is just $\geq$ relation restricted to A.

in general: $x \longrightarrow y$ means xRy

Ex. $A = \{1, 2, 3, 4\}$

$$S = \{(1,1), (1,3), (3,1), (3,3), (2,2), (2,4), (4,2), (4,4)\}$$

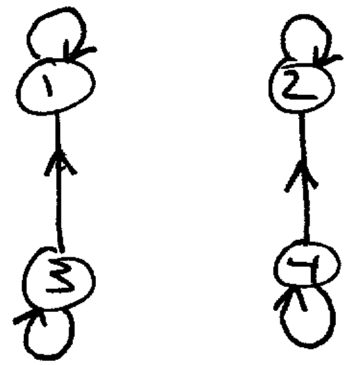
digraph:



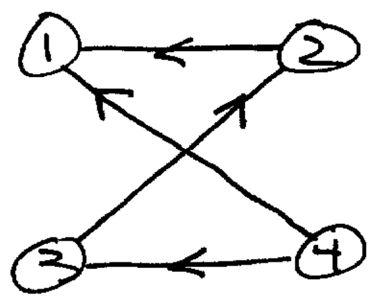
This is "same parity" relation, i.e. $\equiv_2$

since relations are sets, we can apply set operations to them

$R \cap S$:



$R - S$:



<u>Exercise</u>

Draw digraph representations of:

$A \times A$, $(A \times A) - S$, $(A \times A) - R$

$R \cup S$, $R \oplus S$, $S - R$.

(11.2) Properties of relations

Defn

we call a relation $R \subseteq A \times A$

- reflexive iff $\forall x \in A$

$$xRx$$

- symmetric iff $\forall x, y \in A$

$$xRy \Longrightarrow yRx$$

- transitive iff $\forall x, y, z \in A$

$$xRy \wedge yRz \Longrightarrow xRz$$

- antisymmetric iff $\forall x, y \in A$

$$xRy \wedge yRx \Longrightarrow x = y$$

Recall common relations on $\mathbb{Z}$:

Relation	Ref.	Symm.	trans.	anti-symm
$<$	No	no	yes	yes
$\leq$	yes	no	yes	yes
$>$	no	no	yes	yes
$\geq$	yes	no	yes	yes
$=$	yes	yes	yes	yes
$\neq$	no	yes	no	no
$\mid$	yes	no	yes	yes
$\equiv_m$	yes	yes	yes	no

Exercise:

Prove all 32 entries above

- $\equiv_m$ is transitive

$$\left.\begin{array}{l} x \equiv_m y \Rightarrow m \mid x-y \\ y \equiv_m z \Rightarrow m \mid y-z \end{array}\right\} \Rightarrow m \mid (x-y)+(y-z) = x-z$$

$$\Rightarrow x \equiv_m z.$$

- $|$ is <u>not</u> symmetric.

we must show: $\forall x,y : x|y \rightarrow y|x$

is false, i.e. that

$$\sim(\forall x,y : x|y \rightarrow y|x)$$

$$= \exists x,y : x|y \wedge y\nmid x$$

Let $x=2, y=4$. Then $2|4$ but $4\nmid 2$.

- $<$ is antisymmetric.

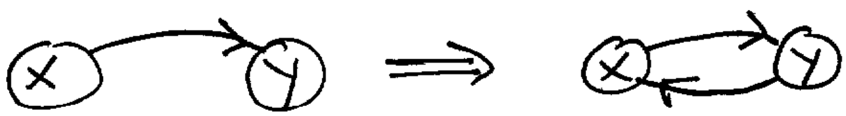
must show: $\forall x,y : x<y \wedge y<x \Rightarrow x=y$

The hypothesis is a contradiction i.e. always false, hence the implication is always true (vacuously).

Digraph interpretation:

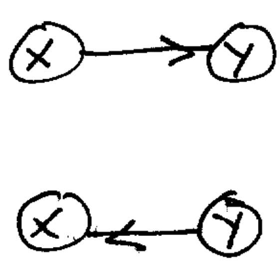
- reflexive: xRx

- symmetric: $xRy \Rightarrow yRx$

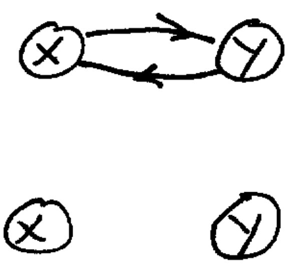


equivalently:

not Possible

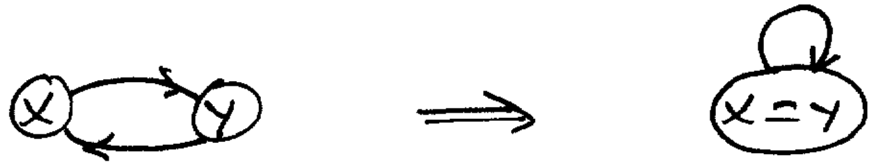


Possible



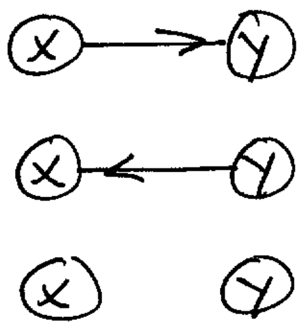
- transitive: $xRy \wedge yRz \Rightarrow xRz$

- antisymmetric: $xRy \wedge yRx \Rightarrow x=y$



equivalently: given $x \neq y$

<u>Possible</u>



<u>not possible</u>

Matrix representation of a relation

<u>Ex.</u> $A = \{1, 2, 3, 4\}$

$$R = \{(1,1), (2,2), (2,1), (3,3), (3,2), (3,1), (4,4), (4,3), (4,2), (4,1)\}$$

$$M(R) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$M(R) = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$$S = \{ (1,1), (1,3), (3,1), (3,3), (2,2), (2,4), (4,2), (4,4) \}$$

$$M(S) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

- How many relations on $A = \{1, 2, 3, 4\}$?

  ans $= 2^{16}$.

- how many reflexive relations on A ?

  ans $= 2^{12}$