cse 16 8-2-22

SETs : student eval. of teaching

- open: Thur 8/4 7:00 am
- close: Tue 8/9 11:59 pm

Theorem (Binomial Theorem)

Let $x, y \in \mathbb{R}$

$P(n)$

$$\forall n \geq 0 : \boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

we use induction form IIb, also use Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad (0 < k < n)$$

Proof:

I. $P(0)$ says $(x+y)^0 = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k$

i.e. $(x+y)^0 = \binom{0}{0} x^0 y^0$, i.e. $1=1$ ✓.

II b. $\forall n>0: P(n-1) \to P(n)$

Let $n>0$. Assume

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k .$$

must show

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

Then

$$(x+y)^n = (x+y)\cdot(x+y)^{n-1}$$

$$= (x+y)\cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind.} \\ \text{hyp.} \end{array} \right.$$

$$= x \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^{k} + y \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^{k}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^{k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^{k+1}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^{k} + \sum_{k=1}^{n} \binom{n-1}{k-1} x^{n-\cancel{1}-(k-\cancel{1})} y^{(k-\cancel{1})+\cancel{1}}$$

$$= \binom{n-1}{0} x^{n} y^{0} + \sum_{k=1}^{n-1} \binom{n-1}{k} x^{n-k} y^{k}$$

$$+ \sum_{k=1}^{n-1} \binom{n-1}{k-1} x^{n-k} y^{k} + \binom{n-1}{n-1} x^{0} y^{n}$$

$$= \binom{n-1}{0} x^{n} y^{0} + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] x^{n-k} y^{k} + \binom{n-1}{n-1} x^{0} y^{n}$$

$$= \binom{n}{0} x^{n} y^{0} + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^{k} + \binom{n}{n} x^{0} y^{n}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k} .$$

Result follows for all $n \geq 0$ by 1st P.M.I.

Recursion

Many sequences can be defined efficiently by recursion

- define a set of base objects.
- specify rule(s) for constructing new objects from existing objects.

Ex. Fibonacci sequence

$$\begin{cases} F_1 = 1 \\ F_2 = 1 \\ \underbrace{F_n = F_{n-1} + F_{n-2}}_{\text{recurrence formula}} \quad (n \geq 3) \end{cases}$$

$$(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, \ldots)$$
$$= (1, 1, 2, 3, 5, 8, 13, 21, \ldots)$$

Rule

objects defined recursively want to have theorems proved about them by induction.

Ex. $\forall n \geq 1 : \sum_{k=1}^{n} F_{2k-1} = F_{2n}$

$n=1 : \quad F_1 = F_2$ ✓

$n=2 : \quad F_1 + F_3 = F_4$ ✓

$n=3 : \quad F_1 + F_3 + F_5 = F_6$ ✓

I. $P(1)$ and $P(2)$ are true.

II d. $\forall n > 2 : P(1) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$

let $n > 2$. Assume for all k in range $1 \leq k < n$ that

$$\sum_{i=1}^{k} F_{2i-1} = F_{2k}.$$

must show

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}$$

Then

$$\sum_{i=1}^{n} F_{2i-1} = \left(\sum_{i=1}^{n-1} F_{2i-1}\right) + F_{2n-1}$$

$$= F_{2(n-1)} + F_{2n-1} \quad \Big\{ \text{by ind. hyp.}$$

$$= F_{2n-2} + F_{2n-1}$$

$$= F_{2n} . \checkmark$$

$\blacksquare$

<u>Ex.</u>

$$\forall n \geq 1 : \sum_{k=1}^{n} F_k F_{k+1} = \begin{cases} F_{n+1}^2 & (n \text{ odd}) \\ F_n F_{n+2} & (n \text{ even}) \end{cases}$$

<u>check</u>:

$n=1$: $F_1 \cdot F_2 = F_2^2$ ✓

$n=2$: $F_1 F_2 + F_2 F_3 = F_2 F_4$ ✓

$n=3$: $F_1 F_2 + F_2 F_3 + F_3 F_4 = F_4^2$ ✓

Proof

I. done

IIa. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$. Assume

$$\sum_{k=1}^{n} F_k F_{k+1} = \begin{cases} F_{n+1}^2 & (n \text{ odd}) \\ F_n F_{n+2} & (n \text{ even}) \end{cases}$$

must show

$$\sum_{k=1}^{n+1} F_k F_{k+1} = \begin{cases} F_{n+2}^2 & (n \text{ even}) \\ F_{n+1} F_{n+3} & (n \text{ odd}) \end{cases}$$

Then

$$\sum_{k=1}^{n+1} F_k F_{k+1} = \left(\sum_{k=1}^{n} F_k F_{k+1} \right) + F_{n+1} F_{n+2}$$

$$= \left\{ \begin{matrix} F_{n+1}^2 & (n \text{ odd}) \\ F_n F_{n+2} & (n \text{ even}) \end{matrix} \right\} + F_{n+1} F_{n+2}$$

$$= \begin{cases} F_{n+1}^2 + F_{n+1} F_{n+2} & (n \text{ odd}) \\ F_n F_{n+2} + F_{n+1} F_{n+2} & (n \text{ even}) \end{cases}$$

$$= \begin{cases} F_{n+1}\left(F_{n+1} + F_{n+2}\right) & (n \text{ odd}) \\ \left(F_n + F_{n+1}\right) F_{n+2} & (n \text{ even}) \end{cases}$$

$$= \begin{cases} F_{n+1} \cdot F_{n+3} & (n \text{ odd}) \\ F_{n+2}^2 & (n \text{ even}) \end{cases}$$

$$= \begin{cases} F_{n+2}^2 & (n \text{ even}, (n+1) \text{ odd}) \\ F_{n+1} F_{n+3} & (n \text{ odd}, (n+1) \text{ even}). \end{cases}$$

∎

Is there a closed formula for F_n? Yes!!

Define

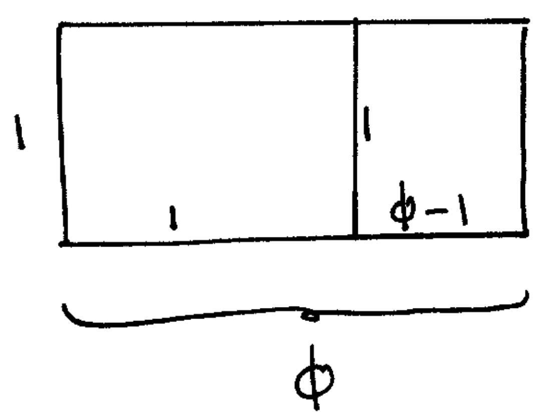
$$\phi_1 = \frac{1+\sqrt{5}}{2} \leftarrow \underline{\text{Golden Ratio}}$$

$$\phi_2 = \frac{1-\sqrt{5}}{2} \leftarrow \underline{\text{Conjugate of G.R.}}$$

note: ϕ_1, ϕ_2 are roots

$$* \qquad x^2 - x - 1 = 0$$

what is special about ϕ_1 ?



$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

$$\phi = \frac{1}{\phi - 1} \rightarrow \phi(\phi - 1) = 1$$

$$\rightarrow \phi^2 - \phi = 1$$

$$\rightarrow \phi^2 - \phi - 1 = 0$$

$\rightarrow \phi$ is a root of $*$.

note: $\boxed{\phi^2 = \phi + 1}$

Theorem

$$\forall n \geq 1 \quad \boxed{F_n = \frac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right)}$$

$P(n)$

use strong ind. form IId, with 2 base cases.

Proof:

I.

✓ $n=1$: $RHS = \frac{1}{\sqrt{5}}(\phi_1 - \phi_2) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$

$= \frac{1}{\sqrt{5}}\left(\frac{\cancel{1} + \sqrt{5} - \cancel{1} + \sqrt{5}}{2}\right) = \frac{1}{\cancel{\sqrt{5}}} \cdot \frac{\cancel{2}\cancel{\sqrt{5}}}{\cancel{2}}$

$= 1 = F_1 = LHS.$

✓ $n=2$: RHS $= \frac{1}{\sqrt{5}}\left(\phi_1^2 - \phi_2^2\right)$

$$= \frac{1}{\sqrt{5}}\left((\phi_1 + \cancel{1}) - (\phi_2 + \cancel{1})\right)$$

$$= \frac{1}{\sqrt{5}}\left(\phi_1 - \phi_2\right)$$

$$= 1 = F_2 = \text{LHS}$$

II d. $\forall n > 2 : P(1) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$

Let $n > 2$. Assume for any k in range $1 \leq k < n$ that

$$F_k = \frac{1}{\sqrt{5}}\left(\phi_1^k - \phi_2^k\right)$$

we must show

$$F_n = \frac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right).$$

so

$$RHS = \frac{1}{\sqrt{5}} \left(\phi_1^n - \phi_2^n \right)$$

$$= \frac{1}{\sqrt{5}} \left(\phi_1^2 \phi_1^{n-2} - \phi_2^2 \phi_2^{n-2} \right)$$

$$= \frac{1}{\sqrt{5}} \left((\phi_1 + 1) \phi_1^{n-2} - (\phi_2 + 1) \phi_2^{n-2} \right)$$

$$= \frac{1}{\sqrt{5}} \left((\phi_1^{n-1} + \phi_1^{n-2}) - (\phi_2^{n-1} + \phi_2^{n-2}) \right)$$

$$= \frac{1}{\sqrt{5}} \left((\phi_1^{n-1} - \phi_2^{n-1}) + (\phi_1^{n-2} - \phi_2^{n-2}) \right)$$

$$= \frac{1}{\sqrt{5}} \left(\phi_1^{n-1} - \phi_2^{n-1} \right) + \frac{1}{\sqrt{5}} \left(\phi_1^{n-2} - \phi_2^{n-2} \right)$$

$$= F_{n-1} + F_{n-2} \quad \left\{ \begin{array}{l} \text{by ind. hyp. with} \\ k = n-1 \;\&\; k = n-2. \end{array} \right.$$

$$= F_n \quad \{ \text{by Fib. rec.}$$

$$= LHS.$$

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