CSE 16 8-10-22 

- SETS now closed.
- be strict when reviewing ind. proofs.
- we will grade ind. proofs strictly on the final.
- Vincent's comments on Ed.
- Regrades (limited time) on Final. (until monday). Note: it is possible to lose points on a regrade.
- Review Problems:

(11.4) # 2, 6
(11.5) # 2, 4, 6
(12.1) # 2, 4, 10, 12
(12.2) # 2, 4, 6, 14, 16
(12.3) # 2

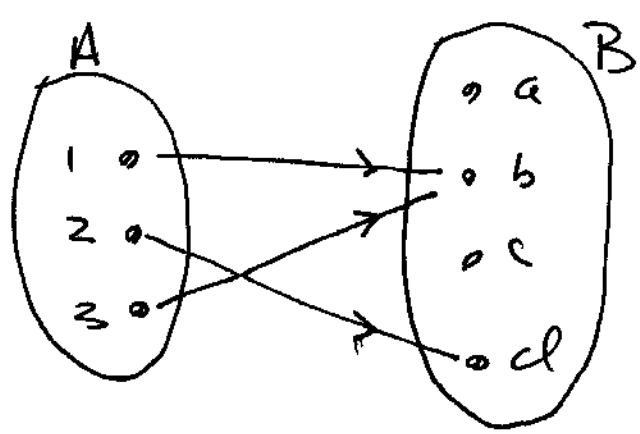
## Alternate definition

A function $f: A \to B$ consists of 3 things

- $A$ : domain
- $B$ : codomain
- rule $f$ : associates to every $a \in A$ a unique $b \in B$, written $b = f(a)$.

**Ex.** $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$,

$f = \{(1,b), (2,d), (3,b)\}$



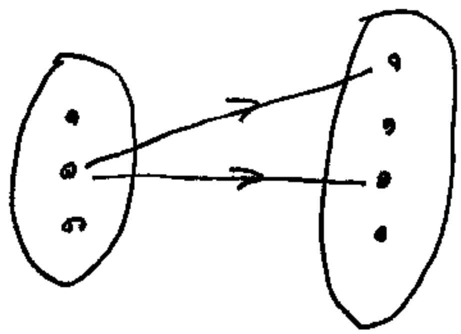
other notation:

$1 \mapsto b$ "1 maps to b"

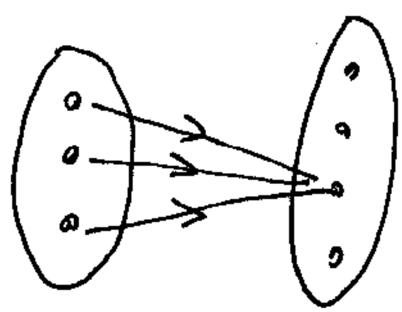
$2 \mapsto d$ "2 " " d"

$3 \mapsto b$ "3 " " b"

<u>note</u>:



not a function since uniqueness is violated

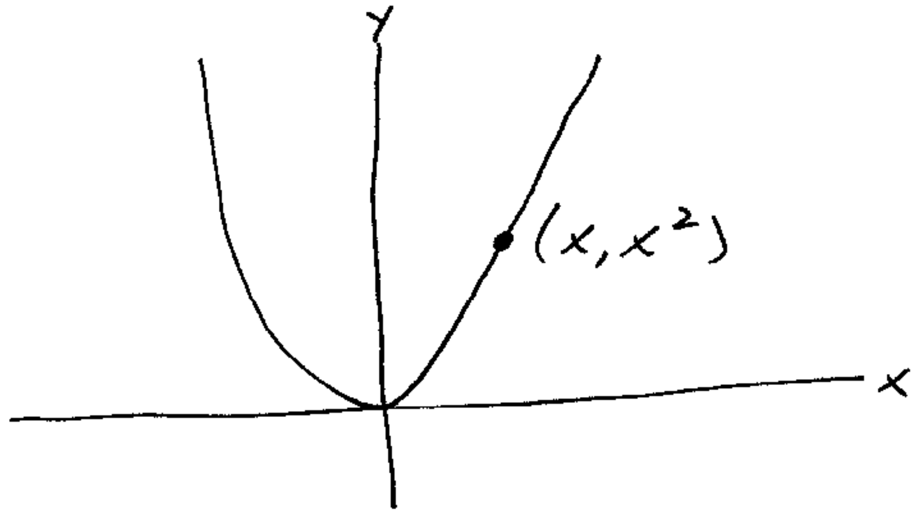


this is fine

<u>Ex</u> $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n^2$

$$\begin{array}{c} \vdots \\ -2 \mapsto 4 \\ -1 \mapsto 1 \\ 0 \mapsto 0 \\ 1 \mapsto 1 \\ 2 \mapsto 4 \\ \vdots \end{array}$$

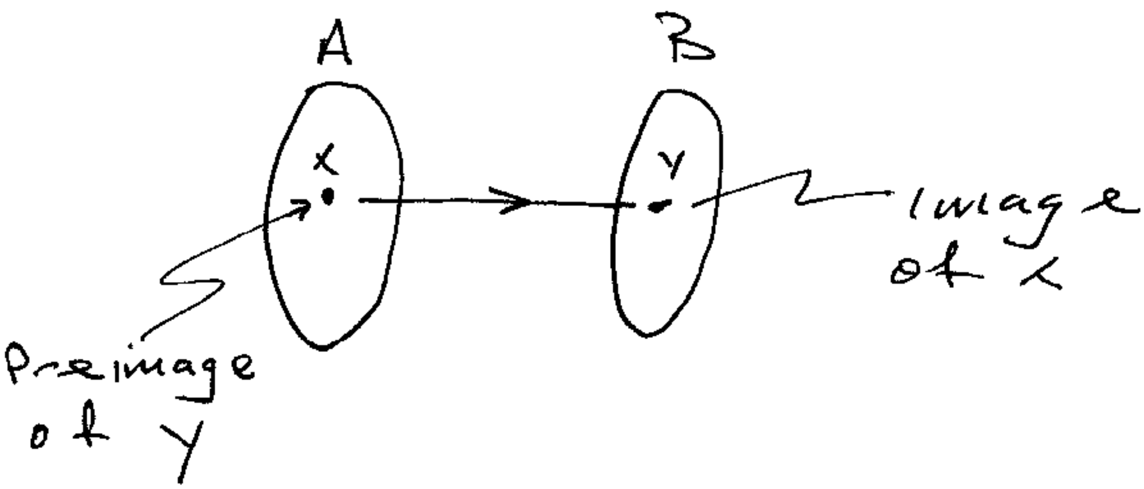
<u>Ex.</u> $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$




<u>Defn</u>

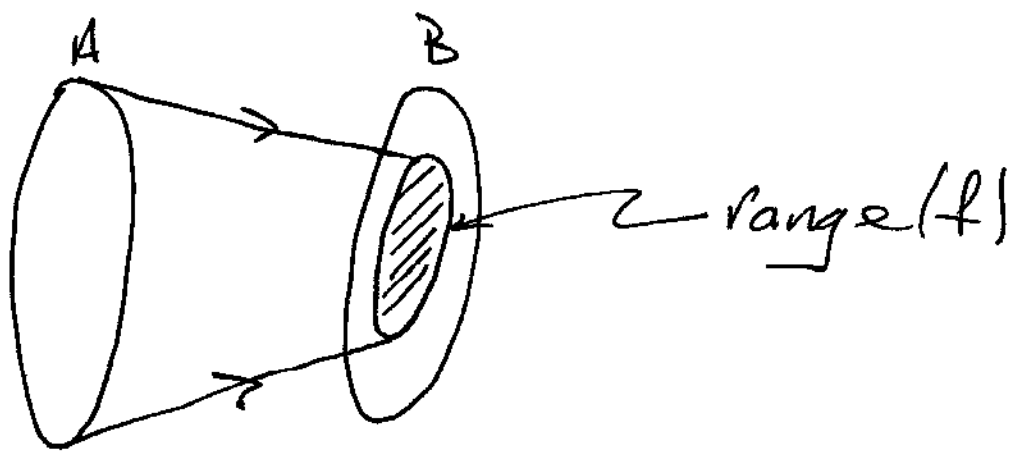
Given $f : A \to B$, $x \in A$, $y \in B$.

- if $y = f(x)$, we say $y$ is <u>the image of $x$ under $f$</u>. Also say $x$ is <u>a preimage of $y$ under $f$</u>.



- The <u>range of $f$</u> is the set

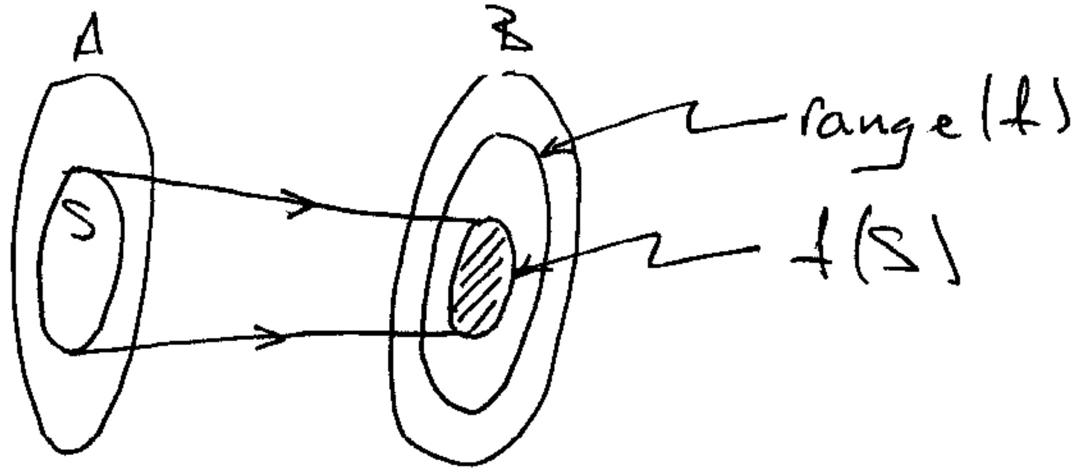
$$\text{range}(f) = \{ y \in B \mid y = f(x) \text{ for some } x \in A \}$$
$$= \{ f(x) \in B \mid x \in A \} \subseteq B$$




- If $S \subseteq A$, the <u>image of $S$ under $f$</u> is the set

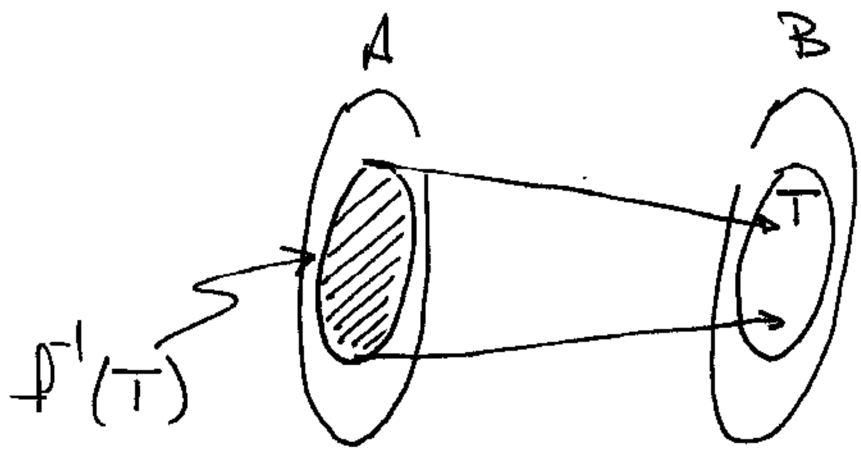
$$f(S) = \{ f(x) \in B \mid x \in S \} \subseteq B$$

<u>note</u>: $\text{range}(f) = f(A)$.




- If $T \subseteq B$, the preimage of $T$ under $f$ is the set

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\} \subseteq A$$



# (12.2) Injective & Surjective fcns.

Defn

A function $f : A \to B$ is called

- <u>injective</u> (or <u>one-to-one</u>) iff

  $\forall x_1, x_2 \in A : \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

  equivalently

  $\forall x_1, x_2 \in A : f(x_1) = f(x_2) \implies x_1 = x_2$

- <u>surjective</u> (or <u>onto</u>) iff

  $\forall y \in B, \exists x \in A : \quad y = f(x)$

  equivalently

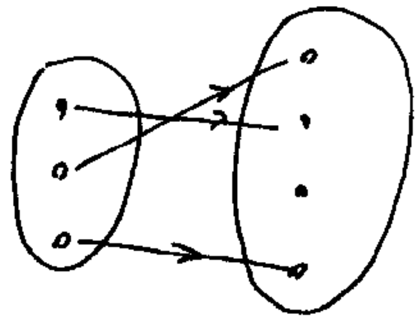
  $\text{range}(f) = B$

- <u>bijective</u> (or a <u>1-1 correspondance</u>) iff

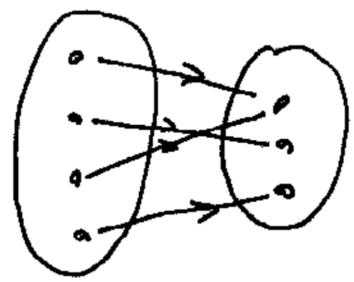
  $f$ is both injective and surjective.

Pictures:

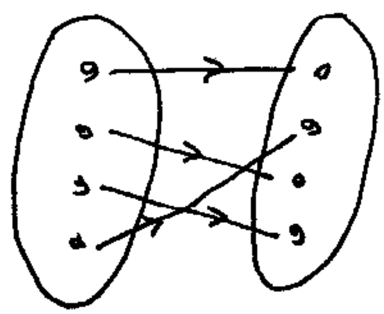
- injective
(not surjective)



- surjective
(not injective)



- bijective



Ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

not injective: $\exists x_1, x_2 \in \mathbb{R} : x_1 \neq x_2 \wedge x_1^2 = x_2^2$

let $x_1 = -1$, $x_2 = 1$, then $-1 \neq 1 \wedge (-1)^2 = 1 = 1^2$.

not surjective: $\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : y \neq x^2$

let $y = -1$. Then $\forall x \in \mathbb{R} : -1 \neq x^2$.

<u>notation</u>: $[0,\infty) = \{ r \in \mathbb{R} \mid 0 \le r \}$

<u>Ex.</u> $f: \mathbb{R} \to [0,\infty)$, $f(x) = x^2$

<u>not injective</u>: same reason

<u>surjective</u>: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : y = x^2$

Let $y \ge 0$ be arbitrary. set $x = \sqrt{y}$.

Then $x^2 = (\sqrt{y})^2 = y$.

<u>Ex.</u> $f: [0,\infty) \to \mathbb{R}$, $f(x) = x^2$

<u>not surjective</u> (same as $1^{st}$ ex.)

<u>injective</u>: $\forall x_1, x_2 \in [0,\infty) : x_1^2 = x_2^2 \Rightarrow x_1 = x_2$

Assume: $x_1^2 = x_2^2$. Then $\sqrt{x_1^2} = \sqrt{x_2^2} \Rightarrow |x_1| = |x_2|$

$\Rightarrow x_1 = x_2$ since $x_1 \ge 0$, $x_2 \ge 0$..

Ex. $f : [0,\infty) \to [0,\infty)$, $f(x) = x^2$

injective : (same reason)

surjective : (same as $2^{nd}$ ex.)

$\therefore$ bijective

Ex. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3x+5$

✓ injective : $\forall x_1, x_2 \in \mathbb{R} : 3x_1+5 = 3x_2+5$
$\Rightarrow x_1 = x_2$.

✓ surjective : $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : y = 3x+5$.

let $y \in \mathbb{R}$ be arbitrary, and let

$$x = \frac{y-5}{3}.$$

Then $g(x) = g\left(\frac{y-5}{3}\right) = \cancel{3}\left(\frac{y-\cancel{5}}{\cancel{3}}\right) + \cancel{5} = y$.

Ex $A = (\mathbb{Z} \times \mathbb{Z}) - \{(0,0)\}$, $B = \mathbb{N}$

$$\gcd : A \to B$$

not injective:

$$\gcd(2,3) = 1 = \gcd(2,5)$$

but $(2,3) \neq (2,5)$.

surjective

let $n \in \mathbb{N}$. Then $\gcd(n,0) = n$.

Theorem

let $A, B$ be finite sets with $|A| = k$ and $|B| = n$. Then the set of fcns $f : A \to B$ is also finite with cardinality $n^k$

<u>notation</u> write $B^A$ for the set of fcns $f : A \to B$.

<u>Thm:</u> $|B^A| = |B|^{|A|}$.

<u>Proof</u>

Let $A = \{1, 2, 3, \dots, k\}$. we can encode a fcn. $f : A \to B$ as a string of length $k$ over alphabet $B$.

$$
\begin{array}{llll}
A & \xrightarrow{\quad f \quad} & B & \underline{\#\text{choices}} \\
1 & \longmapsto & x_1 \in B & n \\
2 & \longmapsto & x_2 \in B & n \\
3 & \longmapsto & x_3 \in B & n \\
 & \vdots & & \\
k & \longmapsto & x_k \in B & n
\end{array}
$$

string: $x_1 x_2 x_3 \cdots x_k$

By the multiplication principle, the # of such fcns (and such strings) is $\underbrace{n \cdot n \cdot n \cdots n}_{k} = n^k$.

Theorem

Let $A, B$ be finite sets with $|A| = k$, $|B| = n$. The # of fcns $f: A \to B$ that are injective is

$$n \cdot (n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!} = P(n,k).$$

Proof (exercise)

## (12.3) Pigeonhole Principle

let $A, B$ be finite, $f: A \to B$.

(1) if $|A| > |B|$, then $f$ is <u>not</u> injective.

(2) if $|A| < |B|$, then $f$ is <u>not</u> surjective.

<u>Theorem</u>

Given $A, B$ finite sets with

$$|A| = |B| = n.$$

The # of bijective fcns $f: A \to B$ is $n!$

<u>Proof</u> (exercise)

(same argument again.)