cse 16 7-7-22

continue Pascals Δ !

<u>observe:</u>

n	
0	$(x+y)^0 = 1$
1	$(x+y)^1 = 1\cdot x + 1\cdot y$
2	$(x+y)^2 = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2$
3	$(x+y)^3 = 1\cdot x^3 + 3\cdot x^2 y + 3\cdot xy^2 + 1\cdot y^3$
4	$(x+y)^4 = 1\cdot x^4 + 4\cdot x^3 y + 6\cdot x^2 y^2 + 4\cdot xy^3 + 1\cdot y^4$
$\vdots$	$\vdots$

note: each term in $(x+y)^n$ is of the form

$$\binom{n}{k}\cdot x^{n-k} y^k \qquad (\text{for } 0 \le k \le n)$$

Binomial Theorem

let $n \geq 0$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Combinatorial Proof:

number each factor of $(x+y)^n$

$$(x+y)^n = \overset{①}{(x+y)} \cdot \overset{②}{(x+y)} \cdot \overset{③}{(x+y)} \cdots \overset{ⓝ}{(x+y)}$$

when this Prod. is fully expanded, and <u>before</u> any like terms are combined each term is a string of len. n over $\{x,y\}$. For instance

$$(x+y)^3 = \overset{①}{(x+y)}\overset{②}{(x+y)}\overset{③}{(x+y)}$$

$$= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

Any terms with the same # of x's and y's can be combined. so,

The coefficient of $x^{n-k}y^k$ <u>after</u> like terms are combined is the number of ways of choosing which k of the n factors will contribute y. This is the # of $\{x, y\}$ strings of len. n containing exactly k y's.

This number is $\binom{n}{k}$. so

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

Call these <u>Binomial Coefficients</u>

Ex. find coeff. of $x^8 y^5$ in $(x+y)^{13}$

ans: $\binom{13}{5} = \dfrac{13!}{5!\,8!}$

$$= \frac{13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}$$

$$= 13 \cdot 11 \cdot 9 = \boxed{1{,}287}$$

Ex. find the coef. of x^5 in $(x-1)^{10}$

$$(-1)^5 \cdot \binom{10}{5} = \cdots\cdots = \boxed{-252}$$

Ex let $x=1,\ y=-1$ in Binomial Thm.

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot (-1)^k$$

$$\therefore \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 .$$

Ex. $x=1, y=1$ in Bin. Thm.

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot 1^k$$

$$\therefore \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Ex. show that

$$9^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^{n-k}$$

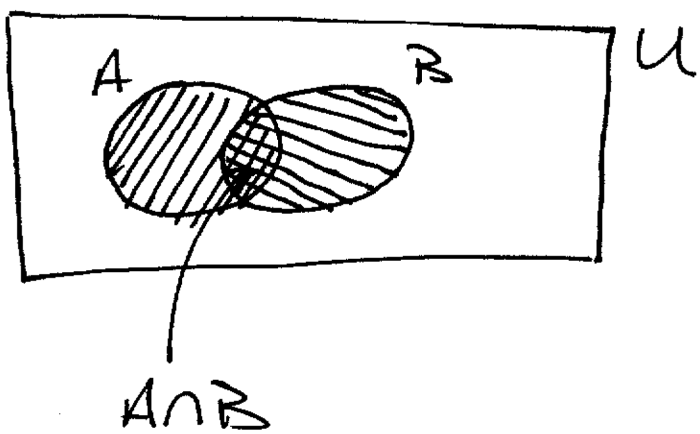
Proof: let $x=10, y=-1$

$$9^n = (10-1)^n = \sum_{k=0}^{n} \binom{n}{k} 10^{n-k} (-1)^k \quad \blacksquare$$

(3.7) Inclusion - Exclusion Principle

if A, B are finite sets, so are $A \cup B$ and $A \cap B$, and

$$|A \cap B| = |A| + |B| - |A \cap B|$$



Ex. how many 5-card poker hands are there?

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \boxed{2,598,960}$$

<u>Ex.</u> how many 5-card Poker hands contain at least one red ace?

let

$A = \{$ 5-card Poker hands cont. Ace $\diamondsuit \}$

$B = \{$ " " " " " Ace $\heartsuit \}$

$A \cap B = \{$ " " " " " Ace $\diamondsuit$ and Ace $\heartsuit \}$

goal: $|A \cup B|$

$$|A| = |B| = \binom{51}{4} = 249,900$$

$$|A \cap B| = \binom{50}{3} = 19,600$$

$$\therefore |A \cup B| = |A| + |B| - |A \cap B|$$

$$= 2(249,900) - 19,600$$

$$= \boxed{480,200}$$

Another way:

$\overline{(A \cup B)} = \{$ " " " " with no red are $\}$

$|\overline{A \cup B}| = \binom{50}{5}$

exercise

$\therefore |A \cup B| = \binom{52}{5} - \binom{50}{5} = \ldots\ldots = ?$

PIE for 3 sets A, B, C

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

PIE for 4 sets A, B, C, D

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$

$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C|$$

$$- |B \cap D| - |C \cap D|$$

$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$

$$+ |B \cap C \cap D|$$

$$- |A \cap B \cap C \cap D|$$

General PIE for n sets:

$$A_1, A_2, A_3, \ldots, A_n$$

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{i=1}^{n} |A_i|$$

$$- \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$

$$\vdots$$

$$+ (-1)^{n+1} \cdot \left|\bigcap_{k=1}^{n} A_k\right|$$

<u>Ex.</u>

Suppose $|A| = |B| = |C| = 20$, $|A \cup B \cup C| = 42$, $|A \cap B| = |A \cap C|$ and $B \cap C = \phi$. find $|A \cap B|$.

Let $x = |A \cap B|$.

$\therefore \; 42 = 3 \cdot 20 - x - x - 0 + 0$

$2x = 60 - 42 = 18$

$\therefore \; \boxed{x = 9}$

(3.8) Multisets

An unordered collection in which multiple memberships are allowed is called a multiset.

Defn
The cardinality of a multiset A, denoted $|A|$, is its number of members, including repetitions

Ex. $A = [1, 1, 1, 2, 3, 3, 4, 4, 4]$

multiplicity of 1 is : 3

" " 2 is : 1

" " 3 is : 2

" " 4 is : 3

so $|A| = 3 + 1 + 2 + 3 = 9$.

notation:

- sets $\{ \cdots \}$
- lists $(\quad)$
- multisets $[\quad]$

Ex lets Produce multisets based on (i.e. using only elements from)

$$\{1, 2, 3\}$$

#

cardinality 0 : $[\,] = \emptyset$ $\boxed{1}$

cardinality 1 : $[1], [2], [3]$ $\boxed{3}$

cardinality 2 :

$[1,1], [1,2], [1,3], [2,2], [2,3], [3,3]$ $\boxed{6}$

cardinality 3 :

$[1,1,1], [1,1,2], [1,1,3], [1,2,2], [1,2,3]$

$[1,3,3], [2,2,2], [2,2,3], [2,3,3], [3,3,3]$ $\boxed{10}$

<u>Cardinality 4:</u> $\underline{\#}$

[1,1,1,1], [1,1,1,2], [1,1,1,3], [1,1,2,2]
[1,1,2,3], [1,1,3,3], [1,2,2,2], [1,2,2,3] $\boxed{15}$
[1,2,3,3] [1,3,3,3], [2,2,2,2], [2,2,2,3]
[2,2,3,3] [2,3,3,3], [3,3,3,3]

<u>Goal</u>: # of k-element multisets based on an n-element set.

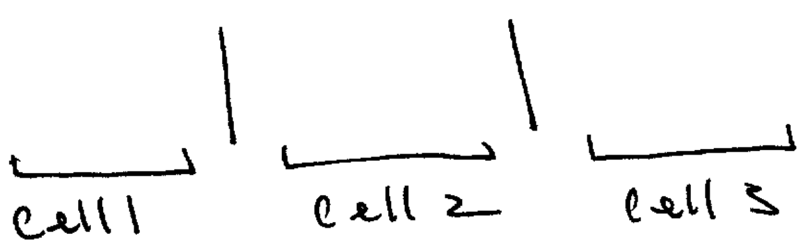
We encode such a multiset as a string over alphabet $\{*, |\}$, i.e. a kind of bit string.

For instance, consider case

$$n=3 \text{ and } k=4$$

multiset		string
[1, 2, 2, 3]	$\longleftrightarrow$	*|**|*
[1, 2, 3, 3]	$\longleftrightarrow$	*|*|**
[1, 3, 3, 3]	$\longleftrightarrow$	*||***
[2, 2, 3, 3]	$\longleftrightarrow$	|**|**
[3, 3, 3, 3]	$\longleftrightarrow$	||****

use $\underset{n}{\underset{\shortparallel}{2=(3}}-1)$ bars and $\underset{k}{\underset{\shortparallel}{4}}$ stars



multiplicity of 1 = #stars in cell 1

" " 2 = " " " 2

" " 3 = " " " 3

<u>note</u>: encoding is invertable.

bit strings of len. 6 containing exactly 2 bars (or exactly 4 stars)

is

$$\binom{6}{2} = \binom{6}{4} = \frac{6 \cdot 5}{2} = 15$$

In general if

$$S = \{x_1, x_2, \dots, x_n\},$$

the multiset of card. k based on S are encoded as

$$\underset{\text{cell } 1}{* \cdots *} \,|\, \underset{\text{cell } 2}{* \cdots *} \,|\, \cdots\cdots \,|\, \underset{\text{cell } n}{** \cdots *}$$

(multiplicity of x_i) = # $*$'s in cell i

$$(\# *\text{'s}) = \sum_{i=1}^{n} \text{mult}(x_i) = k$$

and

$$(\# |\text{'s}) = (\#\text{cells}) - 1 = n - 1$$

General formula:

$$\begin{pmatrix} \#\text{ multisets of card } k \text{ with} \\ \text{base set size } n \end{pmatrix}$$

$$= \binom{k+n-1}{k} = \binom{k+n-1}{n-1}.$$