CSE 16 7-6-22

Ex. list all $P(4,3) = 4 \cdot 3 \cdot 2 = 24$ 3-permutations of $\{1,2,3,4\}$.

123	213	312	412
132	231	321	421
124	214	314	413
142	241	341	431
134	234	324	423
143	243	342	432

Ex (3.4 #7)

find # of 9-digit numbers satisfying

- no zeros : $\{1,2,3,4,5,6,7,8,9\}$
- no repetition
- all odd digits are to the left of all even digits.

odd digits = $\{1, 3, 5, 7, 9\}$

even digits = $\{2, 4, 6, 8\}$

(1) form an odd digit string of len. 5 : $5!$ ways

(2) form an even digit string of len. 4 : $4!$

Then concatenate strings

$$\# \text{strings} = 5! \cdot 4! = (120)(24)$$

$$= \boxed{2,880}$$

<u>Exercise</u>:

Same Problem, but 8-digit, 7-digit and 6-digit strings.

(3.5) Combinations (count subsets)

Defn

A k-combination of a finite set S is a k-element subset of S. i.e an unordered arrangement of k of its elements.

Notation

if $|S| = n$, the # of k-comb. of S is denoted

$$C(n,k) \quad \text{or} \quad \binom{n}{k}$$

read : "n-choose-k"

Ex: $S = \{1, 2, 3\}$

		$C(3,k)$
$k=3$:	$\{1,2,3\}$	1
$k=2$:	$\{1,2\}, \{1,3\}, \{2,3\}$	3
$k=1$:	$\{1\}, \{2\}, \{3\}$	3
$k=0$:	$\emptyset$	1

Theorem

If $0 \le k \le n$ then

$$\binom{n}{k} = C(n,k) = \frac{n!}{k!\,(n-k)!}$$

Proof.

compute $P(n,k)$ in a different way.

recall

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Let S be an n-element set.

new Process

(1) choose a k-element subset of S

$\#$ways $= C(n,k)$

(2) choose a permutation of these

k elements : $\#$ways $= k!$

By the mult. Principle

$$P(n,k) = C(n,k) \cdot k!$$

$$\therefore \quad \frac{n!}{(n-k)!} = C(n,k) \cdot k!$$

$$\therefore \quad C(n,k) = \frac{n!}{k!\,(n-k)!}$$

$\blacksquare$

Ex.

how many bit strings of len. 10 contain exactly 6 ones.

Each bit string of len. 10 ~~encodes~~ with 6 1's encodes a 6-element subset of $\{1,2,3,4,5,6,7,8,9,10\}$.

for instance

$$\underset{1}{1}\ \underset{2}{0}\ \underset{3}{1}\ \underset{4}{0}\ \underset{5}{1}\ \underset{6}{1}\ \underset{7}{0}\ \underset{8}{1}\ \underset{9}{1}\ \underset{10}{0}$$

corresponds to

$$\{1,3,5,6,8,9\}$$

$\therefore$ (# such strings) = (# 6-element ssets of a 10-element set)

$$= \binom{10}{6} = \frac{10!}{6!\,4!} = \frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = \boxed{210}$$

<u>note</u>: a bit str. of len. with 6 1's must 4 0's. Thus

(# bit strs of len 10 with 4 0's)

$= 210$

Also, choosing a 6-sset of a 10-set is equivalent to choosing a 4-sset of a 10-set (its complement)

Thus $\binom{10}{6} = \binom{10}{4}$

choose which 6 positions will be occupied by 1's

choose which 4 positions will be occupied by 0's.

in fact both are: $\dfrac{10!}{6!\,4!}$

Theorem

If $0 \le k \le n$, then

$$\binom{n}{k} = \binom{n}{n-k}$$

Algebraic Proof:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(n-k)! \cdot (n-(n-k))!}$$

$$= \binom{n}{n-k} \qquad \blacksquare$$

Combinatorial Proof:

The LHS counts the number of length n bit strings with exactly k 1's. The RHS counts the number of length n bit strings with exactly (n-k) 0's.

But these 2 sets of bit strings are one and the same.

$\therefore$ LHS = RHS, i.e

$$\binom{n}{k} = \binom{n}{n-k}. \qquad \blacksquare$$

Correspondence between ssets & bitstrings

$S = \{1, 2, 3\}$

k	subset		bitstring	#1's
0	$\emptyset$	$\longleftrightarrow$	000	0
1	$\{1\}$	$\longleftrightarrow$	100	1
	$\{2\}$	$\longleftrightarrow$	010	
	$\{3\}$	$\longleftrightarrow$	001	
2	$\{1,2\}$	$\longleftrightarrow$	110	2
	$\{1,3\}$	$\longleftrightarrow$	101	
	$\{2,3\}$	$\longleftrightarrow$	011	
3	$\{1,2,3\}$	$\longleftrightarrow$	111	3

<u>Exercise</u>: write correspondence for $n = 4$.

(3.6) Binomial Theorem

using : $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!} \quad (0 \le k \le n)$

we compute

n	
0	$\binom{0}{0} = 1$
1	$\binom{1}{0} = 1, \binom{1}{1} = 1$
2	$\binom{2}{0} = 1, \binom{2}{1} = 2, \binom{2}{2} = 1$
3	$\binom{3}{0} = 1, \binom{3}{1} = 3, \binom{3}{2} = 3, \binom{3}{3} = 1$
4	$\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1$
5	$\binom{5}{0} = 1, \binom{5}{1} = 5, \binom{5}{2} = 10, \binom{5}{3} = 10, \binom{5}{4} = 5, \binom{5}{5} = 1$
$\vdots$	

Pascals Δ :

n													
0							1						
1						1		1					
2					1		2		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		15		6		1
⋮													

Properties of Pascal's Δ:

(1) 1's on sides

(2) each interior entry is the sum of the neighboring entries on row above.

Theorem (Pascal's identity)

for $1 \le k \le n$:

$$(*) \quad \boxed{\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}}$$

equivalently, for $0 < k < n$:

$$(**) \quad \boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Algebraic Proof of (*)

$$\text{RHS} = \binom{n}{k-1} + \binom{n}{k}$$

$$= \frac{n!}{(k-1)!\,(n-(k-1))!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{k \cdot n!}{k\,(k-1)!\,(n-k+1)!} + \frac{(n-k+1)\cdot n!}{k!\,(n-k+1)\cdot(n-k)!}$$

$$= \frac{k \cdot n!}{k!\,(n-k+1)!} + \frac{(n-k+1) \cdot n!}{k!\,(n-k+1)!}$$

$$= \frac{(\cancel{k} + n - \cancel{k} + 1)\, n!}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)!}{k!\,((n+1)-k)!}$$

$$= \binom{n+1}{k}$$

$$= \text{LHS} \quad \blacksquare$$

Combinatorial Proof (**)

Let $S = \{$ len. n bit strings with k 1's $\}$

let $A = \{ x \in S \mid x$ ends with $1 \}$

$B = \{ x \in S \mid x$ ends with $0 \}$

clearly $S = A \cup B$ and $A \cap B = \emptyset$, so by addition Principle:

$$|S| = |A| + |B|.$$

obviously $|S| = \binom{n}{k}$.

- each $x \in A$ can be obtained by choosing a bit str. of len. $(n-1)$ with $(k-1)$ 1's, and appending 1

  $$\# \text{ways} = \binom{n-1}{k-1}$$

  $$\therefore |A| = \binom{n-1}{k-1}$$

- each $x \in B$ can be obtained by choosing a bit str. of len. $(n-1)$ containing k 1's, and appending 0:

$$\# \text{ways} = \binom{n-1}{k}$$

$$\therefore |B| = \binom{n-1}{k}$$

Therefore

$$\binom{n}{k} = |S| = |A| + |B|$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$\blacksquare$