CSE 16 7-5-22

Chap 3: counting & combinatorics

(3.1) Lists

Defn

A list (or sequence) is an ordered collection of objects, where repetition is allowed.

notation: (1, 3, 2, 2, 1)

empty list: ()

the objects in a list are called entries, elements, or terms.

The number of elements in a list is its length.

Defn

a string is a sequence of symbols from some alphabet (set), written without commas or Parentheses.

Ex. list of len. 5

$$(1, 3, 2, 2, 1)$$

string of len. 5 over $\{1, 2, 3\}$

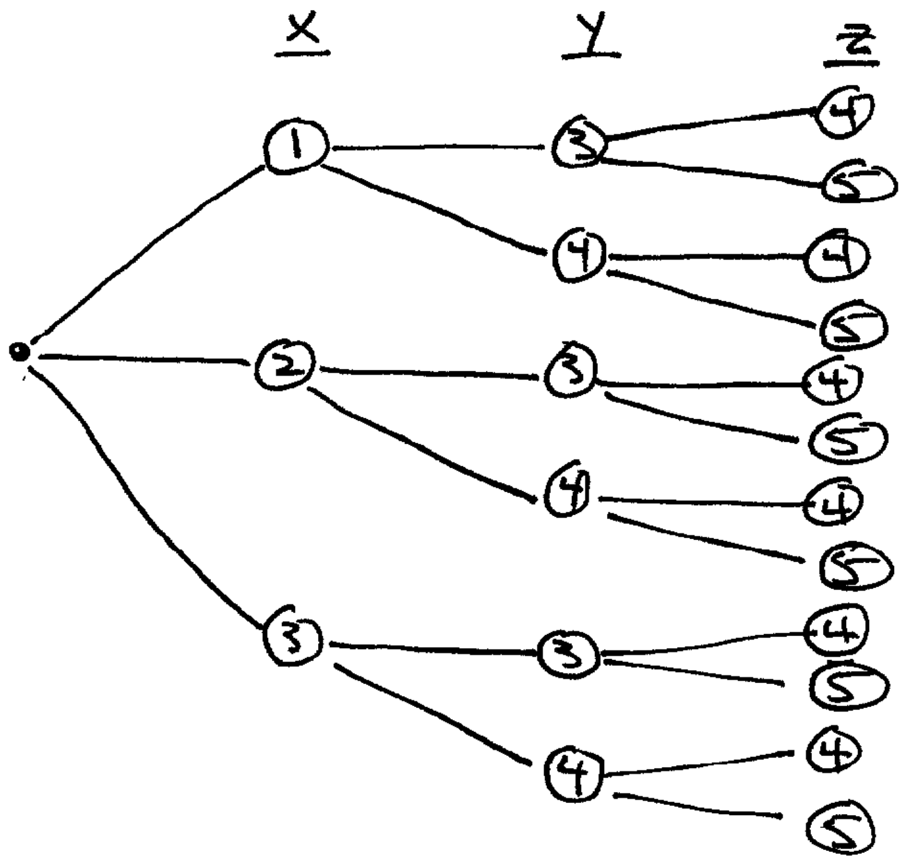
$$13221$$

(3.2) multiplication Principle

Ex. find the number of lists (x, y, z) where $x \in \{1, 2, 3\}$, $y \in \{3, 4\}$, $z \in \{4, 5\}$. i.e

$(1, 3, 4)$, $(3, 3, 5)$, $(2, 4, 4)$, ...

tree diagram



list

(1, 3, 4)
(1, 3, 5)
(1, 4, 4)
(1, 4, 5)
⋮
(3, 4, 5)

there are $\boxed{12}$ such lists.

Generalizing we see

$$\left|\{(x,y,z) \mid (x\in A)\wedge(y\in B)\wedge(z\in C)\}\right| = |A|\cdot|B|\cdot|C|$$

<u>note</u>: this is just $|A\times B\times C| = |A|\cdot|B|\cdot|C|$.

Ex. find # of strings of len. 3 from $\{a, b, c\}$

$$\overset{1}{\underline{\ \cdot\ }}\quad \overset{2}{\underline{\ \cdot\ }}\quad \overset{3}{\underline{\ \cdot\ }}$$

$$\#\text{choices} = 3 \cdot 3 \cdot 3 = \boxed{27}$$

Ex. find # of such strings in which no letter is repeated.

tree diagram:

1	2	3	string
a	b	c	abc
	c	b	acb
b	a	c	bac
	c	a	bca
c	a	b	cab
	b	a	cba

$$\#\text{choices} = 3 \cdot 2 \cdot 1 = \boxed{6}$$

Multiplication Principle

suppose in forming a list $(x_1, x_2, \dots, x_n)$ there are

a_1 choices for x_1
a_2 " " x_2
$\vdots$ $\vdots$
a_i " " x_i
$\vdots$ $\vdots$
a_n " " x_n

Then the # of such lists is

$$a_1 \cdot a_2 \cdot a_3 \cdots a_n$$

note: in general, the number of choices a_i for x_i may depend on the preceding choices (where $i \geq 2$)

$$(\underbrace{x_1, x_2, \dots, x_{i-1},}_{\text{Preceding choices}} x_i, \dots\dots)$$

Ex Bit strings

a <u>bit string</u> are strings over $\{0,1\}$.

The # of bit strings of len. n :

$$\begin{array}{cccccc} \underline{1} & \underline{2} & \underline{3} & \cdots\cdots & \underline{n} & \\ \end{array}$$

$$\#\text{choices} = 2 \cdot 2 \cdot 2 \cdots\cdots 2 = \boxed{2^n}$$

Ex The # of strings of len. n from an alphabet of size a is

$$\begin{array}{cccccc} \underline{1} & \underline{2} & \underline{3} & \cdots\cdots & \underline{n} & \\ \end{array}$$

$$\#\text{choices} = a \quad a \quad a \cdots\cdots a = \boxed{a^n}$$

Ex. The # of such strings with <u>no</u> repetition is

$$\begin{array}{cccccc} \underline{1} & \underline{2} & \underline{3} & & \underline{n} \\ \end{array}$$

$$\#\text{choice} = a \cdot (a-1)\,(a-2) \cdots\cdots (a-n+1)$$

$$= \boxed{a\cdot(a-1)(a-2)\cdots(a-n+1)} = \boxed{\frac{a!}{(a-n)!}}$$

(3.3) Addition & Subtraction Principles

Ex.

How many strings of len. 4 from $\{a,b,c,d,e,f\}$ are there satisfying

- no repeated letters
- must contain a

There are 4 types of such strings

		# of strings
I.	$\underline{a}\ \underset{5}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underset{3}{\underline{\quad}}$	$5\cdot4\cdot3=60$
II.	$\underset{5}{\underline{\quad}}\ \underline{a}\ \underset{4}{\underline{\quad}}\ \underset{3}{\underline{\quad}}$	60
III.	$\underset{5}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underline{a}\ \underset{3}{\underline{\quad}}$	60
IV.	$\underset{5}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underset{3}{\underline{\quad}}\ \underline{a}$	60

(total # of strs) $=$ $\boxed{240}$

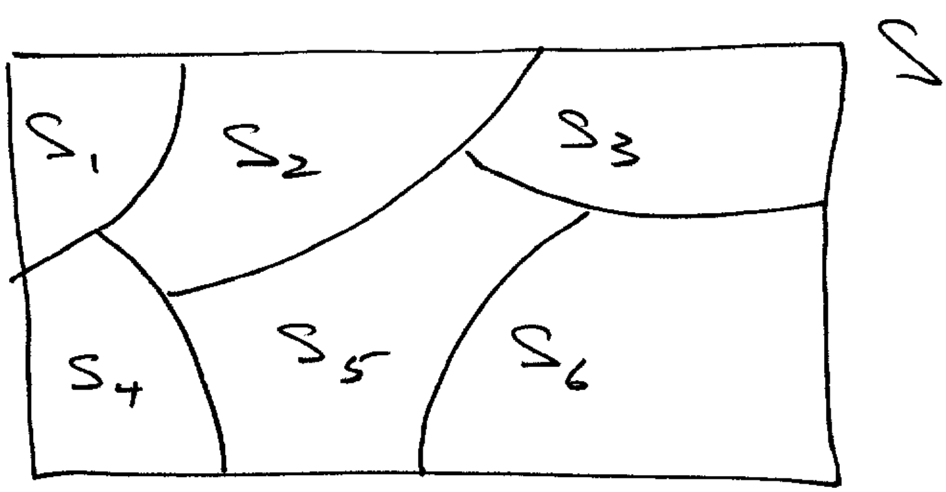
Defn

A Partition of a set S is a collection of subsets $S_1, S_2, \ldots, S_n$ (each $\neq \phi$) such that

(1) $S_i \cap S_j = \phi$ for $1 \leq i < j \leq n$

(Pairwise disjoint)

(2) $\bigcup_{i=1}^{n} S_i = S$



Addition Principle

if $S_1, S_2, \ldots, S_n$ is a Partition of a finite set S, then

$$|S| = |S_1| + |S_2| + \cdots + |S_n|$$

(actually, formula is correct even if some $S_i = \emptyset$.)

Ex

how many 4-digit numbers satisfy

- even
- no 0's, i.e. only $\{1,2,3,4,5,6,7,8,9\}$
- exactly one 5
- exactly one 4, appearing before 5

to be even
must end in $\{2,4,6,8\}$

Partition into 3 subsets

		#
(1)	$\underset{}{\overset{4}{\_}}\ \overset{5}{\_}\ \underset{7}{\overset{\cdot}{\_}}\ \underset{3}{\overset{\cdot}{\_}}$	21
		+
(2)	$\overset{4}{\_}\ \underset{7}{\_}\ \overset{5}{\_}\ \underset{3}{\_}$	21
		+
(3)	$\underset{7}{\overset{\cdot}{\_}}\ \overset{4}{\_}\ \overset{5}{\_}\ \underset{3}{\overset{\cdot}{\_}}$	21
		$\boxed{63}$

Note: for any subset $A \subseteq S$, where $\phi \neq A \neq S$, we have a partition into 2 subset

$$A,\ \overline{A} = S - A$$

Thus $|S| = |A| + |\overline{A}|$. Hence

$$|S-A| = |\overline{A}| = |S| - |A|.$$

(this holds even if $A = 0$ or $A = S$.)

Subtraction Principle

If S is finite and $A \subseteq S$, then

$$|S-A| = |S| - |A|.$$

Ex how many strings of len. 6 from $\{a,b,c,d,e\}$ contain at least 1 d?

let $S = \{$strs. len. 6 over $\{a,b,\ldots,e\}\}$

$A = \{$strs. in S not containing $d\}$

note: $A \subseteq S$ and

$S - A = \{$strs in S containing at least 1 $d\}$

$$|S-A| = |S| - |A|$$

$$= 5^6 - 4^6$$

$$= 15{,}625 - 4{,}096$$

$$= \boxed{11{,}529}$$

(3.4) Permutations

Defn

a <u>Permutation</u> of a finite set S is an arrangement (a list or string with no repetition) of all of its elements.

Ex $S = \{1, 2, 3\}$

Permutations of S:

123, 132, 213, 231, 312, 321

$\#$ Permutations $= \boxed{6}$

Theorem

if $|S|=n$, the # of Permutations of S is

$$n! = n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

we already proved this by the mult. Principle.

Defn

A k-Permutation of a finite set S is an ordered arrangement of k of its elements

Ex. $S=\{1,2,3\}$

		#
$k=3$:	123, 132, 213, 231, 312, 321	6
$k=2$:	12, 21, 13, 31, 23, 32	6
$k=1$:	1, 2, 3	3
$k=0$:	$\emptyset$	1

Notation:

The # of k-permutations of an n-element set is denoted

$$P(n,k)$$

Observe $P(n,0)=1$, $P(n,n)=n!$, and if $k>n$ then $P(n,k)=0$

Thm

$$P(n,k)=\begin{cases} n(n-1)\cdots(n-k+1)=\dfrac{n!}{(n-k)!} & 0\le k\le n \\ 0 & k>n \end{cases}$$

This is the # of strs. of len. k from an n element alphabet, with no repeated letters.