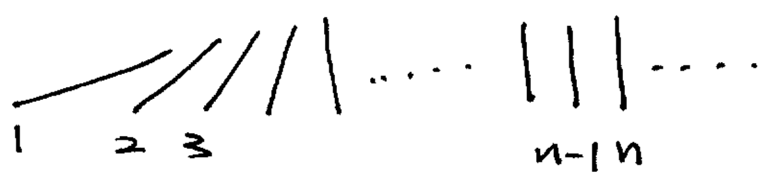
CSE 16 7-28-22

Variations on Induction : $\forall n \geq 1 : P(n)$

I. $P(1)$

IIa. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

IIb. $\forall n > 1 : \underbrace{P(n-1)}_{\text{ind. hyp.}} \rightarrow \underbrace{P(n)}_{\text{ind. concl.}}$



<u>Ex.</u> $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$ (again!) $P(n)$

<u>Proof:</u>

I. same

IIb. $\forall n > 1 : P(n-1) \rightarrow P(n)$

Let $n > 1$ be chosen arbitrarily.

Assume

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} \qquad \text{(ind. hyp.)}$$

we must show

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{(ind. concl.)}$$

so

$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k\right) + n$$

$$= \frac{n(n-1)}{2} + n \qquad \{\text{by ind. hyp.}$$

$$= \frac{n(n-1)+2n}{2}$$

$$= \frac{n(n-1+2)}{2}$$

$$= \frac{n(n+1)}{2}.$$

$\blacksquare$

Exercise:

Go back and re-do all previous examples using form II b.

II a, II b } weak induction or 1^{st} PMI

II c, II d } strong induction or 2^{nd} PMI

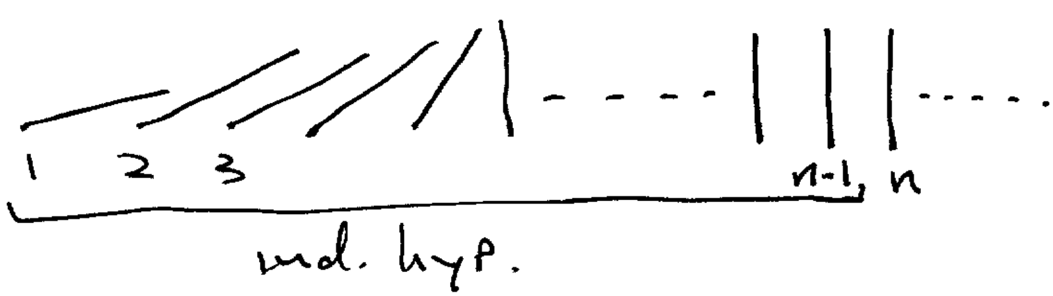
I. $P(1)$ same.

II c. $\forall n \geq 1 : \underbrace{\left(P(1) \wedge P(2) \wedge \dots \wedge P(n)\right)}_{\text{ind. hyp.}} \rightarrow \underbrace{P(n+1)}_{\text{ind. concl.}}$

II d. $\forall n > 1 : \underbrace{\left(P(1) \wedge P(2) \wedge \dots \wedge P(n-1)\right)}_{\text{ind. hyp.}} \rightarrow \underbrace{P(n)}_{\text{ind. concl.}}$

Picture:

II d.



Theorem

Every Positive integer can be expressed as the Product of (zero or more) Primes.

Rmk: this is the existence Part of F.T.A.

Proof:
We show

$\forall n \geq 1$: $\boxed{n \text{ is a Product of Primes}}$ P(n)

I. P(1) is true since 1 is the empty Product

IId. $\forall n > 1 : (P(1) \wedge \ldots \wedge P(n-1)) \Rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume for all k in the range $1 \leq k < n$ that k is a Product of Primes.] ind. hyp.

We must show: n is a prod. of primes] ind. concl.

<u>Case 1</u>: n is prime

In this case n is a product of one prime, itself.

<u>Case 2</u>: n is composite

In this case: $n = a \cdot b$ for some $a, b \in \mathbb{Z}$ with $1 < a < n$ and $1 < b < n$.

By the induction hypothesis both a and b can be written as products of primes, i.e.

$$a = p_1 p_2 \cdots p_k \quad (p_i \text{ prime } 1 \le i \le k)$$

and $$b = q_1 q_2 \cdots q_\ell \quad (q_j \text{ prime } 1 \le j \le \ell)$$

Therefore $n = a \cdot b = (p_1 \cdots p_k)(q_1 \cdots q_\ell)$ is also a product of primes.

Result follows by 2^{nd} PMI. ∎

<u>Theorem</u> (uniqueness part of FTA)
The prime factorization of every $n \geq 1$ is unique (up to order), i.e. if

$$p_1 \cdot p_2 \cdots p_k = n = q_1 \cdot q_2 \cdots q_\ell$$

where $p_1 \leq p_2 \leq \cdots \leq p_k$, $q_1 \leq q_2 \leq \cdots \leq q_\ell$ are primes, then $k = \ell$ and $p_i = q_i$ for $1 \leq i \leq k$.

<u>Proof</u>:

$P(n)$

we show: $\forall n \geq 1$: $\boxed{\text{prime factorization of } n \text{ is unique.}}$

I. $P(1)$ is true since the empty product is unique.

II d. $\forall n > 1 : (P(1) \wedge \ldots \wedge P(n-1)) \rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume the prime factorization of any k in the range $1 \leq k < n$ is unique] ind hyp.

we must show: the prime factorization of n is unique.] ind con.

Suppose

$$* \qquad p_1 p_2 \cdots p_k = n = q_1 q_2 \cdots q_\ell$$

are 2 factorizations of n into primes with $p_1 \leq \cdots \leq p_k$, $q_1 \leq \cdots \leq q_\ell$.

Then $n = (p_1 \cdots p_{k-1}) p_k$. $\therefore$ $p_k \mid n$,

hence $p_k \mid q_1 q_2 \cdots q_\ell$

Aside:

<u>lemma</u>: (ch. 7 #30)

let $a, b, p \in \mathbb{N}$. suppose $a > 1$, $b > 1$ and p prime. If $p \mid a \cdot b$ then $p \mid a$ or $p \mid b$.

<u>corollary</u>

let $a_1, a_2, \ldots, a_k \in \mathbb{Z}$, and p prime. suppose $p \mid (a_1 \cdot a_2 \cdots a_k)$. Then $p \mid a_i$ for at least one i in range $1 \leq i \leq k$.

By the corollary $p_k \mid q_j$ for some $1 \leq j \leq l$. By re-indexing if necessary we have $p_k \mid q_l$. But q_l has only $\{1, q_l\}$ as divisors since $p_k \neq 1$, we have $p_k = q_l$.

Cancel p_k from both sides of * and define m by

$$p_1 p_2 \cdots p_{k-1} = m = q_1 q_2 \cdots q_{l-1}$$

Since $p_k > 1$, we have $m < n$.

By the ind. hyp. the prime factorization of m is unique, i.e. $k-1 = l-1$ and for $1 \le i \le k-1$ we have $p_i = q_i$. Therefore $k = l$ and

$$p_i = q_i \quad \text{for} \quad 1 \le i \le k$$

Therefore the prime factorization of n is unique. Result follows by 2^{nd} P.M.I. ∎

strong induction with multiple base cases.

to show: $\forall n \geq n_0 : P(n)$

prove

I. $P(n_0), P(n_0+1), \ldots, P(n_1)$

IId. $\forall n > n_1 : \left(P(n_0) \land \ldots \land P(n-1)\right) \rightarrow P(n)$.

Ex.

$P(n)$

$$\forall n \geq 8 : \boxed{\exists\, x, y \geq 0 \text{ such that } n = 3x + 5y}$$

note: $P(1), \ldots, P(7)$ are not all true since at least one of x or y would be negative.

Proof:

I. $P(8)$ is true since: $8 = 3\cdot 1 + 5\cdot 1$

$P(9)$ " " " : $9 = 3\cdot 3 + 5\cdot 0$

$P(10)$ " " " : $10 = 3\cdot 0 + 5\cdot 2$

II d. $\forall n > 10 : (P(8) \wedge \dots \wedge P(n-1)) \rightarrow P(n)$

Let $n > 10$ be arbitrary. Assume for all k in range $8 \le k < n$ that there exist $x_k, y_k \ge 0$ such that

$$k = 3x_k + 5y_k .$$

ind. hyp.

we must show: there exist $x, y \ge 0$ such that

$$n = 3x + 5y$$

ind. cond.

Since $n > 10 \Rightarrow n \geq 11 \Rightarrow n-3 \geq 8$, the induction hypothesis provides $x', y' \geq 0$ s.t.

$$n-3 = 3x' + 5y'$$

Let $x = x'+1$ and $y = y'$. Then

$$\begin{aligned} n &= (n-3) + 3 \\ &= (3x' + 5y') + 3 \\ &= 3(x'+1) + 5y' \\ &= 3x + 5y. \end{aligned}$$

Result follows for all $n \geq 8$ by the 2nd PMI. ∎