CSE 16 7-27-22

**Ex.** Let $x \in \mathbb{R}$, $x \neq 1$. Show

$$\forall n \geq 1 : \boxed{\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}} \leftarrow P(n)$$

**Proof.**

I. $P(1)$ says $\displaystyle\sum_{k=0}^{0} x^k = \frac{x^1 - 1}{x - 1}$, i.e.

$$x^0 = \frac{x-1}{x-1}, \text{ i.e. } 1 = 1. \checkmark$$

II. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$ be arbitrary. Assume

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} \quad \text{(ind. hyp.)}$$

we must show

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} \qquad \text{(ind. conc.)}$$

observe

$$\sum_{k=0}^{n} x^k = \left(\sum_{k=0}^{n-1} x^k\right) + x^n$$

$$= \left(\frac{x^n-1}{x-1}\right) + x^n \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \frac{x^n-1+x^n(x-1)}{x-1}$$

$$= \frac{\cancel{x^n}-1+x^{n+1}-\cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1}-1}{x-1} \, .$$

Result follows for all $n \geq 1$ by P.M.I. $\blacksquare$

Rmks

- always state: Let $n \geq 1$ ....
- always ~~state the~~ induction hypothesis explicitely (i.e. not "assume $P(n)$").
- always state induction conclusion explicitely.
- always state the point(s) at which the induction hypothesis is used.

Ex. $\forall n \geq 1$ : $\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$ $P(n)$

Proof.

I. $P(1)$ says $\sum_{k=1}^{1} k = \frac{1(1+1)}{2}$, i.e.

$1 = \frac{1 \cdot 2}{2}$, i.e. $1 = 1$, which is true.

II. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$ be arbitrary. Assume

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{(ind. hyp.)}$$

we must show

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2} \qquad \text{(ind. concl.)}.$$

observe

$$\sum_{k=1}^{n+1} k = \left( \sum_{k=1}^{n} k \right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}.$$

Result follows for all $n$ by the P.M.I. ∎

Ex. $\forall n \geq 1: \boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2} \longleftrightarrow P(n)$

I. $P(1)$ says $\sum_{k=1}^{1} k^3 = \left(\frac{1(1+1)}{2}\right)^2$, i.e.

$1^3 = \left(\frac{1\cdot 2}{2}\right)^2$, i.e. $1^3 = 1^2$, i.e. $1 = 1$ ✓.

II. $\forall n \geq 1: P(n) \rightarrow P(n+1)$

Let $n \geq 1$. Assume

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad \text{(ind. hyp.)}$$

We must show

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 \quad \text{(ind. Cond.)}.$$

Observe

$$\sum_{k=1}^{n+1} k^3 = \left(\sum_{k=1}^{n} k^3\right) + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \left\{\begin{array}{l}\text{by the}\\ \text{ind. hyp.}\end{array}\right.$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$

$$= \frac{(n+1)^2(n^2+4(n+1))}{4}$$

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2 .$$

Result follows for all $n \geq 1$ by P.M.I. ∎

<u>note:</u> $P(n)$ is sometimes untrue for a finite # of initial values of $n$, i.e

$$P(1), P(2), \ldots, P(n_0 - 1)$$

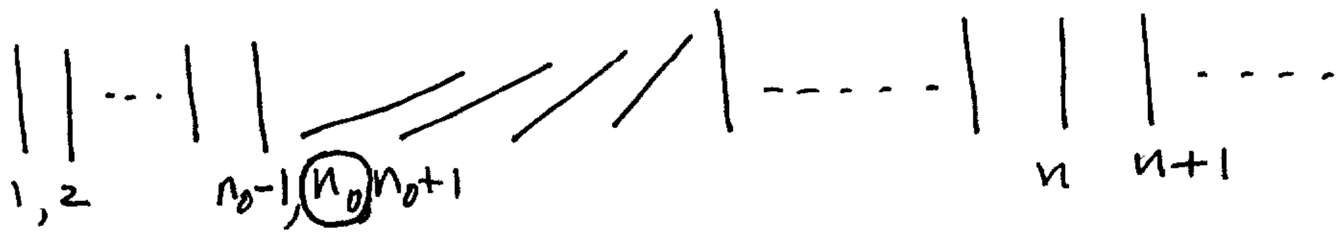
are all false.

To Prove: $\forall n \geq n_0 : P(n)$

we show

I. $P(n_0)$ is true.

II. $\forall n \geq n_0 : P(n) \rightarrow P(n+1)$




Ex. $\forall n \geq 4 : \boxed{5n + 8 < n^2 + 4n + 1}$ ← $P(n)$

(check: $P(1)$, $P(2)$, $P(3)$ are false.)

Proof:

I. $P(4)$ is $5 \cdot 4 + 8 < 4^2 + 4 \cdot 4 + 1$, i.e.

$28 < 33$, which is true.

II. $\forall n \geq 4 : P(n) \Rightarrow P(n+1)$

Let $n \geq 4$ be arbitrary. Assume

$$5n+8 < n^2+4n+1 \quad \text{(ind. hyp.)}$$

we must show

$$5(n+1)+8 < (n+1)^2+4(n+1)+1 \quad \text{(ind. concl.)}$$

observe

$$
\begin{aligned}
5(n+1)+8 &= (5n+8)+5 \\
&< (n^2+4n+1)+5 \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind hyp.} \end{array} \right. \\
&= n^2+4n+6 \\
&< n^2+4n+6+2n \quad \left\{ \begin{array}{l} \text{since} \\ n \geq 4 \Rightarrow 2n \geq 8 > 0 \end{array} \right. \\
&= n^2+6n+6 \\
&= (n^2+2n+1)+(4n+4)+1 \\
&= (n+1)^2+4(n+1)+1 .
\end{aligned}
$$

Result follows for all $n \geq 4$ by P.M.I.

Ex. (ch. 10 #17)

Let $A_1, A_2, A_3 \ldots$ be an infinite sequence of subsets of $S$.
show

$$\forall n \geq 1 : \boxed{\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}} \leftarrow P(n)$$

Rmk: This is generalized DeMorgan Law.

Proof:

I. $P(1)$ says $\overline{\bigcap_{k=1}^{1} A_k} = \overline{A_1} = \bigcup_{k=1}^{1} \overline{A_1}$ ✓

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

Let $n \geq 1$ be arbitrary. Assume

$$\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k} \qquad \text{(ind. hyp.)}$$

must show:

$$\overline{\bigcap_{k=1}^{n+1} A_k} = \bigcup_{k=1}^{n+1} \overline{A_k}$$

so

$$\overline{\bigcap_{k=1}^{n+1} A_k} = \overline{\left(\bigcap_{k=1}^{n} A_k\right) \cap A_{n+1}}$$

$$= \overline{\left(\bigcap_{k=1}^{n} A_k\right)} \cup \overline{A_{n+1}} \quad \left\{ \text{by the 2-set Demorgan law} \right.$$

$$= \left(\bigcup_{k=1}^{n} \overline{A_k}\right) \cup \overline{A_{n+1}} \quad \left\{ \text{by the ind. hyp.} \right.$$

$$= \bigcup_{k=1}^{n+1} \overline{A_k}$$

<u>Exercise</u> (ch 10 #18)

$$\forall n \geq 1 : \overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$$

**Exercise**

Let $P_1, P_2, \ldots, P_n, \ldots$ be an infinite sequence of statements.

**notation** :

$$\bigvee_{k=1}^{n} P_k = P_1 \vee P_2 \vee \cdots \vee P_n$$

$$\bigwedge_{k=1}^{n} P_k = P_1 \wedge P_2 \wedge \cdots \wedge P_n$$

show

- $$\sim\left(\bigwedge_{k=1}^{n} P_k\right) = \bigvee_{k=1}^{n} (\sim P_k)$$

- $$\sim\left(\bigvee_{k=1}^{n} P_k\right) = \bigwedge_{k=1}^{n} (\sim P_k)$$

Recall:

__Well ordering Property of $\mathbb{N}$__

any non-empty subset of $\mathbb{N}$ contains a least element.

__Theorem__ (P.M.I.)

Given any Propositional function $P(n)$ on $\mathbb{N}$, the statement

$$\left[P(1) \wedge \forall n \left(P(n) \rightarrow P(n+1)\right)\right] \rightarrow \left(\forall n \, P(n)\right)$$

is true.

__Proof:__

Assume $P(1)$ and $\forall n \left(P(n) \rightarrow P(n+1)\right)$ are both true. We must show

$$\forall n \, P(n)$$

is true.

Let $S \subseteq \mathbb{N}$ be defined as

$$S = \{ n \in \mathbb{N} \mid \sim P(n) \}$$

i.e. $S$ is the set of pos. ints. for which $P(n)$ is false. It will be sufficient to show $S = \phi$, for then $P(n)$ is true for all $n \in \mathbb{N}$.

Assume (to get a X) that $S \subseteq \mathbb{N}$ is not empty: $S \neq \phi$.

By the W.O.P of $\mathbb{N}$, $S$ contains a least element, call it $m$.

Thus $\boxed{m \in S}$ and $m-1 \notin S$.

Since $P(1)$ is true, $1 \notin S$, so $m \neq 1$, so $m \geq 2$ and $m-1 \geq 1$

$\therefore$ $m-1 \in \mathbb{N}$.

note since $m-1 \notin S$ we have that

$$\boxed{P(m-1) \text{ is true.}}$$

Since $\forall n\,(P(n) \rightarrow P(n+1))$ is true, we have, by substituting $n = m-1$, that

$$\boxed{P(m-1) \longrightarrow P(m) \text{ is true}}$$

By modus ponens we have that

$$P(m) \text{ is true.}$$

Thus $\boxed{m \notin S}$, contradicting that $\boxed{m \in S}$.

This -X- shows our assumption was false, hence $S = \emptyset$.

This completes the proof. $\blacksquare$