CSE 16 7-26-22

Thm

There exists $x, y \in \mathbb{R} - \mathbb{Q}$ such that $x^y \in \mathbb{Q}$.

Lemma $\log_2(9)$ is irrational.

[note: $2^{\log_2(9)} = 9$]

Proof of lemma:

Assume $\log_2(9) \in \mathbb{Q}$. Then

$$\log_2(9) = \frac{a}{b}$$

where $a, b \in \mathbb{Z}$ $(b \neq 0)$.

Thus

$$2^{\log_2(9)} = 2^{\frac{a}{b}}$$

$$\therefore \quad 9 = 2^{\frac{a}{b}}$$

$$\therefore \quad 9^b = \left(2^{\frac{a}{b}}\right)^b$$

$$\therefore \quad 9^b = 2^a$$

But the LHS is odd while RHS is even. This ※ shows that no such a, b exist. Hence

$$\log_2(9) \in \mathbb{R} - \mathbb{Q} .$$

∎

Proof of thm

Let $x=\sqrt{2}$ and $y=\log_2(9)$.

We've shown x and y are irrational.

However

$$x^y = \sqrt{2}^{\log_2(9)} = \sqrt{2}^{\log_2(3^2)}$$

$$= \sqrt{2}^{2\cdot\log_2(3)}$$

$$= \left(\sqrt{2}^{2}\right)^{\log_2(3)}$$

$$= 2^{\log_2(3)}$$

$$= 3 \in \mathbb{Q}. \qquad \blacksquare$$

Exercise

- Try $x=\sqrt{3}$, $y=\log_3(25)$
- Try $x=\sqrt[3]{2}$, $y=\log_2(27)$

Theorem

Let $a, b \in \mathbb{N}$. Then there exist $n, m \in \mathbb{Z}$ such that

$$\gcd(a,b) = an + bm$$

Proof:

Let $S = \{ax + by \mid x, y \in \mathbb{Z}\}$. Note $S \cap \mathbb{N}$ contains only positive integers

$\therefore S \cap \mathbb{N} \subseteq \mathbb{N}$, and $S \cap \mathbb{N}$ contains a least element, by the W.O.P. for $\mathbb{N}$.

Note $S \cap \mathbb{N}$ contains both a $(x=1, y=0)$ and b $(x=0, y=1)$, so is non-empty.

Let d be the least positive element in S.

We will show that $d = \gcd(a,b)$.

By defn of S, there exist $n, m \in \mathbb{Z}$ such that

$$d = an + bm$$

we must show

(i) $d \mid a$ and $d \mid b$

and (ii) if $e \mid a$ and $e \mid b$, then $e \leq d$.

<u>Pf. of (i)</u>

to show $d \mid a$, use division algorithm.

$$a = q \cdot d + r \quad (0 \leq r < d)$$

$$\begin{aligned} \therefore \quad r &= a - qd \\ &= a - q(an + bm) \\ &= a(1 - qn) + b(-qm) \end{aligned}$$

we see r has the form $ax+by$, and hence $r \in S$. But

$$0 \le r < d.$$

If $r>0$, we would have $0<r<d$. This would contradict the definition of d is the <u>least</u> positive element in S. Therefore $r=0$.

$$\therefore \quad a = qd$$

$$\therefore \quad d \mid a$$

A similar argument (exercise) shows that $d \mid b$. $\therefore$ (i) holds.

<u>Proof of (ii)</u>

Suppose $e \mid a$ and $e \mid b$. By a previous lemma $e \mid (an+bm) = d$ $\therefore$ $e \le d$.

Proving (ii).

we've shown

$$\gcd(a,b) = d = an + bm,$$

as required. $\blacksquare$

<u>Note</u>

we've shown alittle more, i.e. that $\gcd(a,b)$ is charactized as the least positive integer comb. of a and b.

Chap. 10 : Mathematical Induction.

Ex.

$$1 = 1^2$$
$$1+3 = 2^2$$
$$1+3+5 = 3^2$$
$$1+3+5+7 = 4^2$$
$$\vdots$$

we might guess

$$1+3+5+\cdots+(2n-1) = n^2$$

for any $n \in \mathbb{N}$. other notation

$$\forall n \in \mathbb{N}: \quad \sum_{k=1}^{n} (2k-1) = n^2$$

How to prove all infinitely many instances of this statement?

Let $P(n)$ be a propositional function with universe $\mathbb{N}$. Suppose we wish to prove

$$* \qquad \forall n \in \mathbb{N} : P(n)$$

i.e. $P(n)$ is true for all $n \in \mathbb{N}$.

A Proof of * by Induction has 2 steps.

I. Base

Prove that $P(1)$ is true.

II. Induction

Prove that $\forall n \geq 1 : P(n) \rightarrow P(n+1)$.

i.e. let $n \geq 1$ be an arbitrary positive integer. Assume $P(n)$ is true. Show as a consequence that $P(n+1)$ is also true.

once I and II are complete we conclude * is true.

The statement $P(n)$ is called the <u>induction hypothesis</u>.

<u>Rmk</u>.

Assuming $P(n)$ in the induction step is <u>not</u> circular reasoning, since we only assume stmt. is true for one n.

Imagine a proof like this:

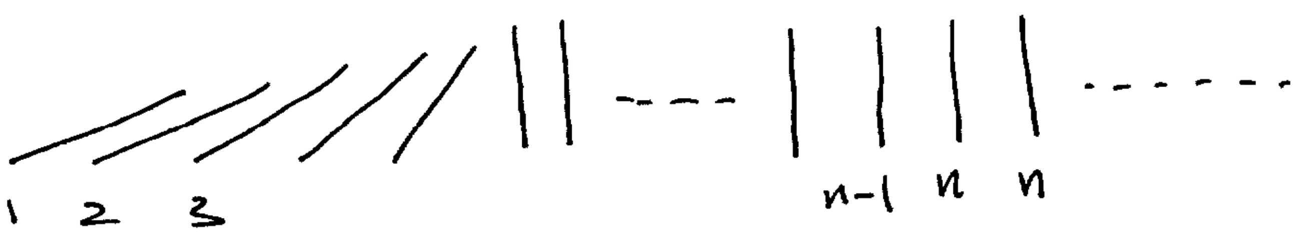
$P(1)$	by I.
$P(1) \Rightarrow P(2)$	by II with $n=1$
$P(2)$	by <u>modus Ponens</u>: $(p \wedge (p \Rightarrow q)) \Rightarrow q$ is a tautology.

$P(2) \Rightarrow P(3)$	by II, $n=2$
$P(3)$	by M.P. again.
$P(3) \Rightarrow P(4)$	by II, $n=3$
$P(4)$	by M.P.

$\vdots$

The problem with this sequence of statements is that it is infinite. A proof is supposed to be finite.

Domino Analogy



let $P(n)$ = "the n^{th} domino falls" we wish to prove

$$* \qquad \forall n \geq 1 : P(n)$$

which says: "all dominos fall"

we prove:

I. $P(1)$ = "the 1^{st} domino falls"

II. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

which says: "if any domino falls, then the next domino falls."

we conclude from I & II that * "all dominos fall".

The validity of this proof is based on

(PMI)

<u>Theorem</u> (Principle of Mathematical Induction)

For any propositional function $P(n)$ on universe $\mathbb{N}$, the following is true

$$[P(1) \wedge \forall n (P(n) \to P(n+1))] \to (\forall n : P(n))$$

we will prove this using the W.O.P of $\mathbb{N}$

<u>Ex.</u> $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} (2k-1) = n^2} \leftarrow P(n)$

I. $P(1)$ is $\sum_{k=1}^{1} (2k-1) = 1^2$, i.e.

$2 \cdot 1 - 1 = 1^2$, i.e. $1 = 1$ which is true.

II. $\forall n \geq 1 : P(n) \to P(n+1)$.

Let $n \geq 1$ be chosen arbitrarily

Assume (to this n) that

$$\sum_{k=1}^{n} (2k-1) = n^2, \qquad \text{(ind. hypothesis)}$$

we must show that

$$\sum_{k=1}^{n+1} (2k-1) = (n+1)^2 \qquad \text{(ind. conclusion)}.$$

we have

$$\begin{aligned}
\sum_{k=1}^{n+1} (2k-1) &= \left(\sum_{k=1}^{n} (2k-1)\right) + (2(n+1)-1) \\
&= n^2 + (2n+1) \quad \left\{ \begin{array}{l}\text{by the} \\ \text{ind. hyp.}\end{array} \right. \\
&= n^2 + 2n + 1 \\
&= (n+1)^2.
\end{aligned}$$

By the P.M.I, we conclude

$$\forall n \geq 1 : \sum_{k=1}^{n} (2k-1) = n^2 .$$