CSE 16 7-13-22

Proof of P.P.:

The average # of obj. per box is $\frac{n}{k}$

we cannot have all boxes containing below the average. # of obj.

$\therefore$ at least one box contains $\geq \frac{n}{k}$ objects

$\therefore$ " " " " " $\geq \lceil \frac{n}{k} \rceil$ objects.

This proves (a)

Similarly not all boxes contain above the average number $\frac{n}{k}$ of objects.

$\therefore$ some box has # obj $\leq \frac{n}{k}$

$\therefore$ .. .. .. # obj $\leq \lfloor \frac{n}{k} \rfloor$.

This proves (b)

Ex

find the smallest n s.t. in any group of n people, at least 10 were born in the same month.

P.P. (a) says: if n objects (people) are placed in 12 boxes (months), then some box (month) contains $\lceil \frac{n}{12} \rceil$ (or more) objects (people).

we seek the smallest n for which

$$\left\lceil \frac{n}{12} \right\rceil = 10$$

$$\therefore \quad 9 < \frac{n}{12} \leq 10$$

$$\therefore \quad 12 \cdot 9 < n \leq 12 \cdot 10$$

$$\therefore \quad 108 < n \leq 120$$

$$\therefore \quad \boxed{n = 109}$$

Ex.

Suppose 8 numbers are chosen from $\{1, 2, 3, \ldots, 14\}$. Show that at least 2 of the 8 numbers must sum to 15.

We could check all $\binom{14}{8} = 3003$ 8-element subsets of $\{1, 2, \ldots, 14\}$ for this property.

Instead, we form 7 labeled boxes

(1,14) (2,13) (3,12) (4,11) (5,10) (6,9) (7,8)

Each of the 8 chosen numbers is to be placed in the box with that number in its label.

By P.P.(a) some box must contain $\lceil \frac{8}{7} \rceil = 2$ numbers, and those 2 numbers add to 15.

Generalization

what is the smallest n such that in any n-subset of $\{1, 2, 3, \ldots, 2k\}$ contains 2 numbers that sum to $2k+1$.

<u>answer</u>: $n = k+1$

want $\left\lceil \frac{n}{k} \right\rceil = 2$

<u>Exercise</u>:
Try to generalize this further.

(3.10) Combinatorial Proofs

Ex. (3.6 #10)

show $\quad k \cdot \binom{n}{k} = n \binom{n-1}{k-1}$

Proof

Imagine we have n players from which we must choose a team of size k, and one person on the team to be captain. We do this in 2 ways.

I. (1) choose k people from n : $\binom{n}{k}$

(2) choose 1 of the k as capt. : k

By the mult. principle

$$\# \text{ways} = k \cdot \binom{n}{k}$$

II. (1) choose 1 of n as capt. : n

(2) from remaining $(n-1)$ people, choose $(k-1)$ teammates : $\binom{n-1}{k-1}$

By mult. principle

$$\# \text{ways} = n \cdot \binom{n-1}{k-1}$$

Since both expressions count all pairs of the form

(captain, team),

we have : $k\binom{n}{k} = n\binom{n-1}{k-1}$. $\blacksquare$

<u>Remark</u>
more abstractly, both sides count

$$\{(x,A) \mid x \in A \subseteq S \text{ and } |A| = k\}$$

where $|S| = n$.

Ex (generalization of prev., and of 3.10 #3)

show

$$\binom{n}{k}\cdot\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$$

where $0 \le r \le k \le n$.

Proof:
Let $|S| = n$. We count the set

$$\{(A,B) \mid A \subseteq B \subseteq S,\ |A| = r,\ |B| = k\}$$

I. (1) choose $B \subseteq S$ with $|B| = k$.
   #ways $= \binom{n}{k}$
   (2) choose $A \subseteq B$ with $|A| = r$
   # ways $= \binom{k}{r}$

By mult. Principle, the above set has size: $\binom{n}{k}\cdot\binom{k}{r} = LHS$

II. (1) choose $A \subseteq S$ with $|A| = r$

$$\#\text{ways} = \binom{n}{r}$$

(2) choose $(k-r)$ elements from $S - A$. form B by adding the chosen elements to A so that $|B| = r + (k-r) = k$

$$\#\text{ways} = \binom{n-r}{k-r}.$$

By mult. Principle # Pairs (A, B) is $\binom{n}{r} \cdot \binom{n-r}{k-r} = RHS$.

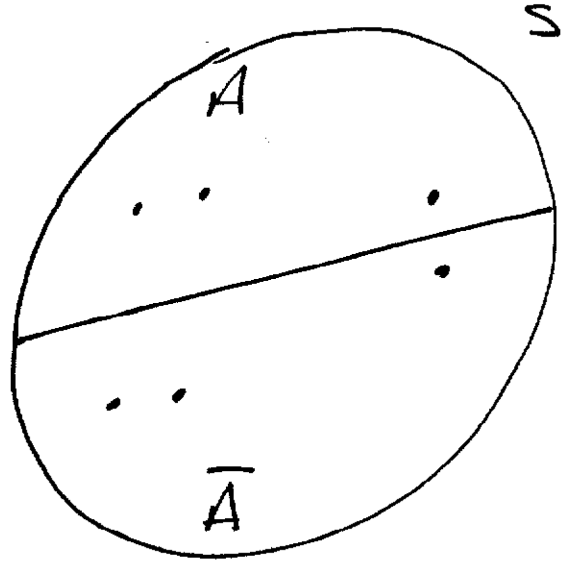
Thus $LHS = RHS$. $\blacksquare$

Ex. (3.10 #5)

$$\binom{2n}{2} = 2 \cdot \binom{n}{2} + n^2$$

Proof:

Let $|S| = 2n$, and let $A \subseteq S$ with $|A| = n$. Then $\bar{A} = A - S$ has n elements also.



$\{x, y\} \subseteq S$

I. the # of 2-element subsets of S is obviously $\binom{2n}{2} = LHS$.

II. to count 2-subsets of S another way, notice they fall into 2 categories

- Pick $\{x,y\} \subseteq S$ with either $x, y \in A$ or $x, y \in \bar{A}$.

(1) choose A or $\bar{A}$: # ways $= 2$
(2) choose $\{x,y\} \subseteq A$ (or $\bar{A}$) : # ways $= \binom{n}{2}$

subsets $\{x, y\} \subseteq S$ of this type $= 2 \cdot \binom{n}{2}$

- Pick $\{x,y\} \subseteq S$ where x, y belong to different halves: $x \in A$, $y \in \bar{A}$ or $x \in \bar{A}$, $y \in A$.

  (1) choose $x \in A$: # ways $= n$
  (2) choose $y \in \bar{A}$: # ways $= n$

  so # of $\{x,y\} \subseteq S$ of this type is $n \cdot n = n^2$

By the addition principle, the # of 2-subsets of S is

$$2 \cdot \binom{n}{2} + n^2 = \text{RHS}.$$

Since both sides count the same thing, we have

$$\binom{2n}{2} = 2\binom{n}{2} + n^2 .$$

$\blacksquare$

Exercise (3.10 #6)

Show that

$$\binom{3n}{3} = n^3 + 6n\binom{n}{2} + 3\binom{n}{3}$$

Ex. (P. 109)

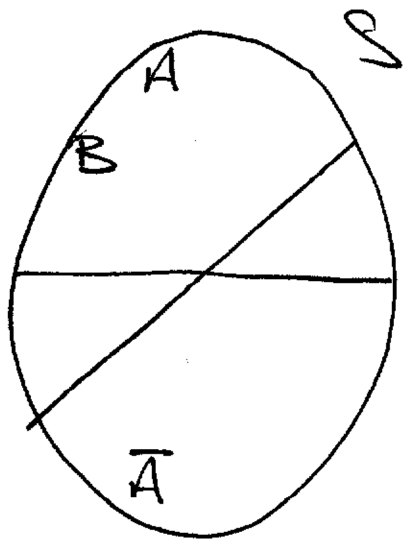
Show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
n=3 \rightarrow & & & \multicolumn{7}{c}{\boxed{1^2 + 3^2 + 3^2 + 1^2}} & = 20 \checkmark & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
2n=6 \rightarrow 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\vdots & & & & & & & & & & & & \ddots
\end{array}
$$

Proof:

Let $|S| = 2n$, and $A \subseteq S$ with $|A| = n$.
Then $|\bar{A}| = n$ also.



consider $\{B \subseteq S \mid |B| = n\}$. obviously this set has cardinality $\binom{2n}{n} = RHS$.

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