Theorem

The number of k-element multisets based on n-element set is the number of strings over $\{*, |\}$ of length $k+n-1$ containing k $*$'s (or equiv. $n-1$ bars), i.e.

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

earlier seq. of examples!

$k = 0, 1, 2, 3, 4, \ldots$

$n = 3$

base set: $\{1, 2, 3\}$

$k=0: \binom{0+3-1}{0} = \binom{2}{0} = 1$

$k=1: \binom{1+3-1}{1} = \binom{3}{1} = 3$

$k=2: \binom{2+3-1}{2} = \binom{4}{2} = \frac{4 \cdot 3}{2} = 6$

$k=3: \binom{3+3-1}{3} = \binom{5}{3} = \frac{5 \cdot 4}{2} = 10$

$k=4: \binom{4+3-1}{4} = \binom{6}{4} = \frac{6 \cdot 5}{2} = 15$

<u>Ex.</u> (3.19 P. 99 of B.O.P)

have an unlimited supply of Red, Blue and Green marbles. In how many ways can you select 20 marbles?

Each collection of 20 marbles is a 20-multiset based on the 3-set $\{R, B, G\}$. so $k=20$, $n=3$ and

$$\binom{20+3-1}{20} = \binom{22}{20} = \boxed{231}$$

marble selections.

<u>Ex.</u>

How many non-negative integer solns. to

$$x + y + z = 20$$

{ Diophantine equation

are there?

<u>note</u>: a soln. is a triple (x, y, z) where $x, y, z \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ that satisfies above eqn.

e.g. $(3,10,7)$ and $(5,0,15)$ are solns, while $(3,10,9)$ and $(\frac{1}{2},\frac{5}{2},17)$ are not solns.

Same as prev. example. let

$x = \#$ Red marbles

$y = \#$ Blue "

$z = \#$ Green "

$\therefore \#$ solns $= \binom{22}{20} = 231$

Correspondance

<u>soln</u>	<u>multiset</u>
$(3,10,7)$	$\leftrightarrow [\underbrace{R,R,R}_{3},\underbrace{B,B,B,B,B,B,B,B,B,B}_{10},\underbrace{G,G,G,G,G,G,G}_{7}]$

$\leftrightarrow \underbrace{***}_{3} \mid \underbrace{**********}_{10} \mid \underbrace{*******}_{7}$

<u>string</u>

Ex.

how many int. solns to

$$x + y + z = 20$$

satisfy $x \geq 0$, $y \geq 0$ and $z \geq 5$.

Let $w = z - 5$. then $w \geq 0$ and $z = w + 5$.

so

$$x + y + (z - 5) = 20 - 5 \qquad (1)$$

$$x + y + w = 15 \qquad (2)$$

hence

$$\#\text{solns} = \binom{15 + 3 - 1}{15} = \binom{17}{15} = \boxed{136}.$$

Permutations where some elements are indistinguishable.

Defn

an anagram of a word is an arrangement (permutation) of its letters where repeated letters are indistinguishable.

Ex. LOOK

KLOO	LOOK	OKLO	
LKOO	KOOL	OLKO	$\boxed{12}$
KOLO	OKOL	OOLK	
LOKO	OLOK	OOKL	

How do we count these?

Temporarily mark two O's:

O_1 O_2

$$L O_1 O_2 K$$

permutations of 4 letters is $4!$

To count anagrams, divide out by # of ways of permuting $\{O_1, O_2\}$ i.e. $2!$

$$\text{ans} = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot \cancel{2}}{\cancel{2}} = \boxed{12}$$

<u>Ex</u> Count anagrams of

MISSISSIPPI

Mark repeated letters:

$$M I_1 S_1 S_2 I_2 S_3 S_4 I_3 P_1 P_2 I_4$$

$\#$ permutations $= 11!$

$(\#$ permutations of $\{I_1, I_2, I_3, I_4\}) = 4!$

$($ " " $\{S_1, S_2, S_3, S_4\}) = 4!$

$($ " " $\{P_1, P_2\})) = 2!$

Thus

$$\# \text{anagrams} = \frac{11!}{4! \cdot 4! \cdot 2!} = \boxed{34{,}650}$$

An anagram of MISSISSIPPI can be thought of as a permutation of the 11-multiset

$$[I, I, I, I, M, P, P, S, S, S, S]$$

based on $\{I, M, P, S\}$.

<u>Theorem</u>

the number of permutations of an n-multiset based on a k-set, with multiplicities $p_1, p_2, \ldots, p_k$ is

$$(\#\text{permutations}) = \frac{n!}{p_1!\, p_2! \cdots p_k!}$$

(note necessarily : $p_1 + p_2 + \cdots + p_k = n$)

in mississippi example

$$\frac{11!}{4! \cdot 1! \, 2! \, 4!} = 34{,}650$$

<u>Remark</u>

special notation (not in book)

$$\binom{n}{p_1, p_2, \ldots, p_k} = \frac{n!}{p_1! \, p_2! \cdots p_k!}$$

where $p_1 + \cdots + p_k = n$

called <u>multinomial coefficient</u>.

<u>note</u>: $\underbrace{\binom{n}{p_1, p_2}}_{\text{multinomial coeff.}} = \binom{n}{p_1} = \binom{n}{p_2}$

Theorem

$$\binom{n}{p_1, p_2, \ldots, p_k} = \binom{n}{p_1} \cdot \binom{n-p_1}{p_2} \cdot \binom{n-p_1-p_2}{p_3} \cdots \binom{n-p_1-p_2-\cdots-p_{k-1}}{p_k}$$

Exercise: Prove this

- algebraic: easy!
- combinatorial: easy!

Multinomial Theorem

Given $x_1, x_2, \ldots, x_k \in \mathbb{R}$, $n \geq 0$:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{p_1+p_2+\cdots+p_k = n} \binom{n}{p_1, p_2, \ldots, p_k} x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k}$$

where sum is over all k-tuples $(p_1, \ldots, p_k)$ of non-negative ints satisfying

$$p_1 + p_2 + \cdots + p_k = n$$

Exercise

(try to) Prove this, combinatorially.

Ex.

find # of Pairs (x, y) satisfying

$$0 \le x \le y \le 30$$

where $x \in \mathbb{Z}$, $y \in \mathbb{Z}$.

This is the same as the # of non-negative int. solns to

$$a + b + c = 30$$

How? correspondence

$$(x, y) \longleftrightarrow (a, b, c)$$

where $\begin{cases} a = x \\ b = y - x \\ c = 30 - y \end{cases}$ $\qquad \begin{cases} x = a \\ y = a + b \end{cases}$

<u>note</u>

$0 \le x \to a \ge 0$

$x \le y \to b \ge 0$

$y \le 30 \to c \ge 0$

<u>also:</u>

$a \ge 0 \to 0 \le x$

$b \ge 0 \to x \le y$

$c \ge 0 \to y \le 30$

so

$$\#(x,y) = \#(a,b,c) = \binom{30+3-1}{30} = \binom{32}{30} = \boxed{496}$$

<u>Another way</u>

By sum rule

$$\#(0 \le x \le y \le 30) = \#(0 \le x < y \le 30) + \#(0 \le x \le 30)$$

$$= \binom{31}{2} + 31$$

$$= 465 + 31 = \boxed{496}$$

(3.9) Pigeonhole Principle

Defn

let $x \in \mathbb{R}$.

- $\lfloor x \rfloor$ = greatest int. that is less than or eq. to x. floor of x
- $\lceil x \rceil$ = least int. that is greater than or eq. to x. ceiling of x.

Alternatively: $\lfloor x \rfloor$ & $\lceil x \rceil$ are unique integers satisfying

- $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$
- $\lceil x \rceil - 1 < x \leq \lceil x \rceil$

Equivalently:

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

Pigeonhole Principle (P.P.)

suppose n objects are placed in k boxes. then

(a) at least one box contains $\lceil \frac{n}{k} \rceil$ (or more) objects.

(b) at least one box contains $\lfloor \frac{n}{k} \rfloor$ (or fewer) objects.

Proof:

The average # of objects per box is $\frac{n}{k}$.

we cannot have all boxes below average. ∴ some box must have $\geq \frac{n}{k}$ objects. the # objects in a box is an integer.

Hence some box contains $\geq \lceil \frac{n}{k} \rceil$ objects, proving (a).

Similarly for (b). $\blacksquare$