
EX. Translate: 'every real number has a real cube root' into Predicate logic:

'x is a cube root of y' = $(y^3 = x)$

$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : y^3 = x$ (True!)

notice: this not same as

$\exists y \in \mathbb{R}\ \forall x \in \mathbb{R} : y^3 = x$

This says: 'there is a real number that is a cube root of <u>every</u> real number.'

(2.8) Quantifiers & conditionals

<u>note</u>: in many texts, quantification is implicit

<u>Ex</u>
$$x^2 - y^2 = (x-y)(x+y)$$

what is meant is

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}: x^2 - y^2 = (x-y)(x+y).$$

<u>Ex.</u>

$$\text{-----} \; (x \text{ is a multiple of } 6) \to (x \text{ is even})$$

$\uparrow$

$\forall x \in \mathbb{Z}$

Advice: state quantifiers explicitly.

(2.9) Translation

consider:

'all horses in California are blue'

Ex. $U = \{\text{horses in CA}\}$

$B(x) =$ 'x is blue'

Translation: $\forall x\, B(x)$

Ex. $U = \{\text{horses}\}$

$C(x) =$ 'x lives in CA'

Translation: $\forall x\, (C(x) \rightarrow B(x))$.

Ex. $U = \{\text{living things}\}$

$H(x) =$ 'x is a horse'

Translation: $\forall x\, (H(x) \wedge C(x) \rightarrow B(x))$

<u>note</u>: $T \to P \equiv P$ since

P	$T \to P$
0	0
1	1

so 1st translation:

$$\forall x\, B(x) \equiv \forall x\, (T \to B(x))$$

Thus if $S \subseteq \mathcal{U}$, then

$$\forall x \in \mathcal{U}\ \ ((x \in S) \to P(x))$$

is the same as

$$\forall x \in S : P(x)$$

Ex. $U = \{\text{people}\}$

$L(x,y) =$ 'x loves y'

Translate:

(a) 'everybody loves somebody'

$$\forall x \exists y : L(x,y)$$

(b) 'somebody loves everybody'

$$\exists x \forall y : L(x,y)$$

(c) 'nobody loves everybody'

$$\sim \exists x \forall y : L(x,y)$$

(d) 'there is exactly one person who everyone loves'.

$$\exists y \left(\forall x\, L(x,y) \wedge \forall z \left(z \neq y \rightarrow \exists w \sim L(w,z) \right) \right)$$

(e.) 'there is a person who loves everyone but themself'

$$\exists x \left(\sim L(x,x) \wedge \forall y \left(y \neq x \rightarrow L(x,y) \right) \right)$$

Ex 'no King of England is not the King of France'.

$U = \{\text{People}\}$, $E(x) =$ 'x is King of England', $F(x) =$ 'x is King of France'.

$$\sim \exists x \left(E(x) \wedge \sim F(x) \right)$$

(2.10) Negations

Consider a finite universe

$$U = \{1, 2\}$$

let $P(x)$ be a Prop. fcn. with domain U. Then

- $\forall x\, P(x) = P(1) \wedge P(2)$
- $\exists x\, P(x) = P(1) \vee P(2)$

Thus

$$\begin{aligned} \sim \forall x\, P(x) &= \sim (P(1) \wedge P(2)) \\ &= \sim P(1) \vee \sim P(2) \\ &= \exists x \sim P(x) \end{aligned}$$

Also

$$\sim \exists x\, P(x) = \sim (P(1) \vee P(2))$$

$$= \sim P(1) \wedge \sim P(2)$$

$$= \forall x \sim P(x)$$

In fact, for any prop. fcn. $P(x)$ on any universe $\mathcal{U}$, we have

$$\boxed{\sim \forall x\, P(x) = \exists x \sim P(x)}$$

$$\boxed{\sim \exists x\, P(x) = \forall x \sim P(x)}$$

	true when	false when
$\forall x\, P(x)$	$P(x)$ is true for all $x \in \mathcal{U}$	$P(x)$ is false for at least one $x \in \mathcal{U}$
$\exists x\, P(x)$	$P(x)$ is true for at least one $x \in \mathcal{U}$	$P(x)$ is false for all $x \in \mathcal{U}$.

last example:

'no King of England is not the King of France'.

$$\sim \exists x \, (E(x) \wedge \sim F(x))$$

$$= \forall x \sim (E(x) \wedge \sim F(x))$$

$$= \forall x \, (\sim E(x) \vee \sim\sim F(x))$$

$$= \forall x \, (\sim E(x) \vee F(x))$$

$$= \forall x \, (E(x) \rightarrow F(x))$$

(2.11) Inference Rules

Recall: a proof is a finite sequence of statements, starting with some hypotheses, ending with a conclusion.

Each stmt. that is not a hypothesis, axiom or previously proved fact, must follow from some <u>inference rule</u>.

An inference rule is a tautology.

- <u>Modus Ponens</u>:

$$[(p \to q) \land p] \longrightarrow q$$

p	q	$p \to q$	$(p \to q) \land p$	$((p \to q) \land p) \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

<u>tautology</u>

- **Modus Tollens**

$$[(p \to q) \land \sim q] \to \sim p$$

- **Elimination**

$$[(p \lor q) \land \sim p] \to q$$

- **Conjunction**

$$[(p) \land (q)] \to (p \land q)$$

- **Addition**

$$[p] \to (p \lor q)$$

- **Simplification**

$$[p \land q] \to p$$